

THE GENERATOR RANK FOR C^* -ALGEBRAS

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ABSTRACT. The invariant that assigns to a C^* -algebra its minimal number of generators lacks natural permanence properties. In particular, it may increase when passing to ideals or inductive limits. It is therefore hard to compute this invariant directly.

To obtain a better behaved theory, we not only ask if k generators exist, but also if such tuples are dense. This defines the generator rank, which we show has many of the permanence properties that are also satisfied by other noncommutative dimension theories. In particular, it does not increase when passing to ideals, quotients or inductive limits.

The definition of the generator rank is analogous to that of the real rank, and we show that the latter always dominates the generator rank. The most interesting value of the generator rank is one, which means exactly that the generators form a generic set, that is, a dense G_δ -subset. We compute the generator rank of homogeneous C^* -algebras, which allows us to deduce that certain AH-algebras have generator rank one. For example, every AF-algebra has generator rank one and therefore contains a dense set of generators.

1. INTRODUCTION

The generator problem for C^* -algebras is to determine which C^* -algebras are singly generated. More generally, for a given C^* -algebra A one wants to determine the minimal number of generators, i.e., the minimal k such that A contains k elements that are not contained in any proper sub- C^* -algebra. For a more detailed discussion of the generator problem, we refer the reader to the recent paper by Wilhelm Winter and the author, [TW12], where it is also shown that every unital, separable \mathcal{Z} -stable C^* -algebra is singly generated, see [TW12, Theorem 3.7].

Given a C^* -algebra A , let us denote by $\text{gen}(A)$ the minimal number of *self-adjoint* generators for A , and set $\text{gen}(A) = \infty$ if A is not finitely generated, see [Nag]. The restriction to self-adjoint elements is mainly for convenience. It only leads to a minor variation of the original generator problem, since two self-adjoint elements a, b generate the same sub- C^* -algebra as the element $a + ib$. In particular, A is singly generated if and only if it is generated by two self-adjoint elements, that is, if and only if $\text{gen}(A) \leq 2$. For a compact, metric space X , it is easy to see that $\text{gen}(C(X)) \leq k$ if and only if X can be embedded into \mathbb{R}^k .

The problem with computing the minimal number of self-adjoint generators is that it does not behave well with respect to inductive limits, i.e., in general we do not have $\text{gen}(A) \leq \liminf_n \text{gen}(A_n)$ if $A = \varinjlim A_n$ is an inductive limit. This is unfortunate

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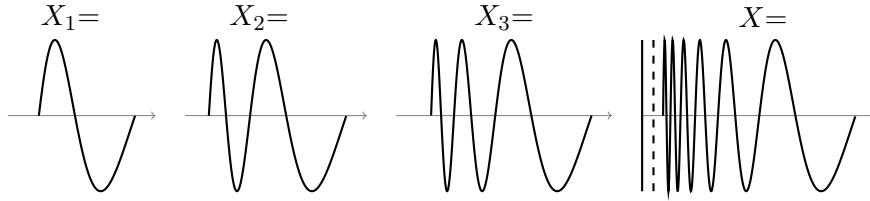
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since many C^* -algebras are given as inductive limits, e.g., AF-algebras or approximately homogeneous algebras (AH-algebras).

To see an example where the minimal number of generators increasing when passing to an inductive limit, let $X \subset \mathbb{R}^2$ be the topologists sine-curve given by:

$$X = \{0\} \times [-1, 1] \cup \{(t, \sin(\frac{1}{t})) \mid t \in (0, 1/2\pi]\}.$$

Then X can be embedded into \mathbb{R}^2 but not into \mathbb{R}^1 , and therefore $\text{gen}(C(X)) = 2$. However, X is an inverse limit of spaces X_n that are each homeomorphic to the interval, i.e., $X_n \cong [0, 1]$. Therefore $C(X) \cong \varprojlim_n C(X_n)$, with $\text{gen}(C(X)) = 2$, while $\text{gen}(C(X_n)) = 1$ for all n . The spaces X and X_1, X_2, X_3 are shown below.



By considering the spaces $X \times [0, 1]$ and $X_n \times [0, 1]$, one obtains an example of singly generated C^* -algebras A_n such that their inductive limit is not singly generated.

To get a better behaved theory, instead of counting the minimal number of self-adjoint generators, we will count the minimal number of “stable” self-adjoint generators. This is the underlying idea of our definition of the generator rank of a C^* -algebra, see Definition 2.2. More precisely, let A_{sa}^k denote the space of self-adjoint k -tuples in A , and let $\text{Gen}_k(A)_{\text{sa}} \subset A_{\text{sa}}^k$ be the subset of tuples that generate A , see Notation 2.1. We say that A has generator rank at most k , denoted by $\text{gr}(A) \leq k$, if $\text{Gen}_{k+1}(A)_{\text{sa}}$ is dense in A_{sa}^{k+1} . This definition is analogous to that of the real rank, see Remark 2.3, and this also explains the index shift of the definition.

Thus, while “ $\text{gen}(A) \leq k$ ” records that $\text{Gen}_k(A)_{\text{sa}}$ is not empty, “ $\text{gr}(A) \leq k - 1$ ” records that $\text{Gen}_k(A)_{\text{sa}}$ is dense. This indicates why the generator rank is usually much larger than the minimal number of self-adjoint generators. The payoff, however, is that the generator rank is much easier to compute.

The paper is organized as follows: In Section 2, we define the generator rank, see Definition 2.2, and we derive some of its general properties. We show that the set of generating tuples, $\text{Gen}_k(A)_{\text{sa}}$, always forms a G_δ -subset of A_{sa}^k , see Proposition 2.7. It follows, that $\text{gr}(A) \leq 1$ if and only if the set of generators in A forms a generic set, i.e., a dense G_δ -subset.

For an inductive limit $A = \varinjlim A_n$, we obtain

$$\text{gr}(A) \leq \liminf_n \text{gr}(A_n),$$

which shows that the generator rank is indeed better behaved than the theory of counting the minimal number of generators. As an immediate consequence, we get that every AF-algebra has generator rank at most one, see Corollary 3.3. Thus, every AF-algebra contains a generic set of generators.

We show that the generator rank does not increase when passing to ideals or quotients, see Theorem 2.14 and Proposition 2.12. We also provide an estimate of the generator

rank of an extension of C^* -algebras in terms of the generator rank of ideal and quotient, see Theorem 2.17.

This shows that the generator rank has many properties which are also satisfied by other “dimension theories” for C^* -algebras, such as the real and stable rank, the nuclear dimension, and the decomposition rank. The concept of a “noncommutative dimension theory” was recently introduced in [Thi11, Definition 2.1] by proposing six axioms that such theories should satisfy. Our results show that the generator rank for separable C^* -algebras satisfies five of these axioms, see Remark 2.16. The remaining axiom would mean that $\text{gr}(A \oplus B) = \max\{\text{gr}(A), \text{gr}(B)\}$, which seems to be surprisingly hard to show.

In Section 3, we study the generator rank on the class of separable C^* -algebras with real rank zero. For such algebras, we show that the remaining axiom holds, i.e., that the generator rank behaves well with respect to direct sums, see Proposition 3.1. We then show that AF-algebras have generator rank at most one, see Corollary 3.3. We proceed by showing the estimate $\text{gr}(A \otimes M_n) \leq \left\lceil \frac{\text{gr}(A)}{n^2} \right\rceil$ for unital C^* -algebras with real rank zero and stable rank one, see Theorem 3.6. Therefore, given such an algebra with finite generator rank, its tensor product with an infinite UHF-algebra has generator rank one. More generally, we show that every separable, real rank zero C^* -algebra that tensorially absorbs a UHF-algebra has generator rank at most one, see Proposition 3.8.

In Section 4, we first compute the generator rank of commutative C^* -algebras as $\text{gr}(C(X)) = \dim(X \times X)$ for a compact, metric space X , see Proposition 4.7. We then compute the codimension of the subspace $\text{Gen}_k(M_n)_{\text{sa}} \subset (M_n)_{\text{sa}}^k$, see Lemma 4.20. This allows us to compute the generator rank of homogeneous C^* -algebras, see Theorem 4.23. In particular, if X is a compact, metric space, and $n \geq 2$, then:

$$\text{gr}(C(X, M_n)) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil.$$

This allows us to show that a unital, separable AH-algebra has generator rank one if it is either simple with slow dimension growth, or when it tensorially absorbs a UHF-algebra, see Corollary 4.30.

Throughout, we will use the following notation. For a C^* -algebra A , we let A_{sa} (resp. A_+ , A^{-1}) denote the set of self-adjoint (resp. positive, resp. invertible) elements in A . We denote by \tilde{A} the minimal unitization of A . By a morphism between C^* -algebras we always mean a $*$ -homomorphism. We write $J \triangleleft A$ to indicate that J is an ideal in A , and by an ideal of a C^* -algebra we understand a closed, two-sided ideal. The primitive ideal space of A will be denoted by $\text{Prim}(A)$. We write M_k for the C^* -algebra of k -by- k matrices $M_k(\mathbb{C})$.

Given $a, b \in A$, and $\varepsilon > 0$, we write $a =_\varepsilon b$ if $\|a - b\| < \varepsilon$. If a, b are positive, then we write $a \ll b$ if $a = ab$, and we write $a \ll_\varepsilon b$ if $a =_\varepsilon ab$. If $F, G \subset A$ are two subsets, and $a \in A$, we write $a \in_\varepsilon G$ if $\text{dist}(a, G) < \varepsilon$, and write $F \subset_\varepsilon G$ if $x \in_\varepsilon G$ for every $x \in F$.

We use bold letters to denote tuples of elements, e.g., $\mathbf{a} = (a_1, \dots, a_k) \in A^k$.

2. THE GENERATOR RANK

In this section, we define the generator rank of a C^* -algebra, see Definition 2.2, in analogy to the real rank, see Remark 2.4. We then prove general properties of the generator rank, in particular that it behaves well with respect to approximation by subalgebras and inductive limits, see Proposition 2.13. We show that the generator rank does not increase when passing to ideals or quotients, see Theorem 2.14 and Proposition 2.12.

We also provide an estimate of the generator rank of an extension of C^* -algebras in terms of the generator rank of ideal and quotient, see Theorem 2.17.

The concept of a noncommutative dimension theory was introduced in [Thi11, Definition 2.1] by proposing six axioms that such theories should satisfy. Our results show that the generator rank for separable C^* -algebras satisfies five of these axioms, see Remark 2.16, and we conjecture that it also satisfies the missing axiom.

Notation 2.1. Let A be a C^* -algebra. Recall that we use bold letters to denote tuples of elements, e.g., $\mathbf{a} = (a_1, \dots, a_k) \in A^k$. We denote by $C^*(\mathbf{a})$ the sub- C^* -algebra of A generated by the elements of \mathbf{a} .

For $k \geq 1$, we write A_{sa}^k for $(A_{\text{sa}})^k$, the space of self-adjoint k -tuples. We denote the set of generating (self-adjoint) k -tuples by:

$$\begin{aligned} \text{Gen}_k(A) &:= \{\mathbf{a} \in A^k \mid A = C^*(\mathbf{a})\}, \\ \text{Gen}_k(A)_{\text{sa}} &:= \text{Gen}_k(A) \cap A_{\text{sa}}^k. \end{aligned}$$

We equip A^k with the usual norm, i.e., $\|\mathbf{a}\| := \max\{\|a_1\|, \dots, \|a_k\|\}$ for a tuple $\mathbf{a} \in A^k$.

Definition 2.2. Let A be a unital C^* -algebra. The **generator rank** of A , denoted by $\text{gr}(A)$, is the smallest integer $k \geq 0$ such that $\text{Gen}_{k+1}(A)_{\text{sa}}$ is dense in A_{sa}^{k+1} . If no such n exists, we set $\text{gr}(A) = \infty$.

Given a non-unital C^* -algebra A , set $\text{gr}(A) := \text{gr}(\tilde{A})$.

Remark 2.3. The definition of the generator rank is analogous to that of the real rank as given by Brown and Pedersen, [BP91]. Let us recall the definition.

Let A be a unital C^* -algebra. One uses the following notation:

$$\begin{aligned} \text{Lg}_k(A) &:= \{\mathbf{a} \in A^k \mid \sum_{i=1}^k a_i^* a_i \in A^{-1}\}, \\ \text{Lg}_k(A)_{\text{sa}} &:= \text{Lg}_k(A) \cap A_{\text{sa}}^k. \end{aligned}$$

The abbreviation “Lg” stands for “left generators”, and the reason is that a tuple $\mathbf{a} \in A^k$ lies in $\text{Lg}_k(A)$ if and only if the elements a_1, \dots, a_k generate A as a (not necessarily closed) left ideal, i.e., $Aa_1 + \dots + Aa_k = A$.

Rieffel introduced the (topological) stable rank of A , denoted by $\text{sr}(A)$, as the smallest integer $k \geq 1$ such that $\text{Lg}_k(A)$ is dense in A^k , see [Rie83, Definition 1.4]. Considering the analogous question for tuples of self-adjoint elements, Brown and Pedersen defined the real rank of A , denoted by $\text{rr}(A)$, as the smallest integer $k \geq 0$ such that $\text{Lg}_{k+1}(A)_{\text{sa}}$ is dense in A_{sa}^{k+1} , see [BP91]. Note the index shift in the definition of the real rank (as opposed to the definition of stable rank). It leads to nicer formulas, e.g., $\text{rr}(C(X)) = \dim(X)$. We use the same index shift in Definition 2.2 since the generator rank is more closely connected to the real rank than to the stable rank, as we will see now.

Remark 2.4. Let A be a C^* -algebra. We may consider the following variant of the generator rank, defined as the smallest integer $k \geq 1$ such that $\text{Gen}_k(A)$ is dense in A^k . Let us denote this value by $\text{gr}'(A)$. Since the generator rank $\text{gr}(A)$ is defined in analogy to the real rank (using tuples of self-adjoint elements), one might expect that the invariant gr' (using tuples of not necessarily self-adjoint elements) has a closer connection to the stable rank.

This is, however, not the case. For instance, while the estimate $\text{rr}(A) \leq \text{gr}(A)$ always holds, see Proposition 2.5, we will below see an example of a C^* -algebra where $\text{sr}(A) \not\leq$

$\text{gr}'(A)$. Moreover, unlike the real and stable rank, the invariants gr and gr' are very closely tied together:

$$\text{gr}'(A) = \left\lceil \frac{\text{gr}(A) + 1}{2} \right\rceil.$$

To prove this formula, consider the map $\Phi: A_{\text{sa}}^{2k} \rightarrow A^k$ that sends $(a_1, \dots, a_{2k}) \in A_{\text{sa}}^{2k}$ to $(a_1 + ia_{k+1}, \dots, a_k + ia_{2k}) \in A^k$. In general, two self-adjoint elements $c, d \in A_{\text{sa}}$ generate the same sub- C^* -algebra as the element $c + id$. It follows $C^*(\mathbf{a}) = C^*(\Phi(\mathbf{a})) \subset A$ for every $\mathbf{a} \in A_{\text{sa}}^{2k}$, and so Φ maps $\text{Gen}_{2k}(A)_{\text{sa}}$ onto $\text{Gen}_k(A)$. Thus, for every $k \geq 1$, $\text{gr}(A) \leq 2k - 1$ if and only if $\text{gr}'(A) \leq k$, from which the formula follows.

Assume now that A is unital. If $\mathbf{a} \in A_{\text{sa}}^k$ generates A as a C^* -algebra, then it also generates A as a left ideal. Indeed, assume p is a polynomial such that $\|1 - p(\mathbf{a})\| < 1$. Then $p(\mathbf{a})$ is invertible, and we denote its inverse by $v \in A$. Write p as a sum of polynomials, $p = \sum_{i=1}^k p_i$, where each p_i is of the form $p_i(\mathbf{x}) = q_i(\mathbf{x}) \cdot x_i$ for some other polynomial q_i . Then:

$$1 = v \cdot p(\mathbf{a}) = \sum_i (v \cdot q_i(\mathbf{a})) a_i,$$

which shows $\mathbf{a} \in \text{Lg}_k(A)_{\text{sa}}$.

Thus, for every $k \geq 1$, the following inclusion holds:

$$\text{Gen}_k(A)_{\text{sa}} \subset \text{Lg}_k(A)_{\text{sa}},$$

which immediately implies Proposition 2.5 below.

The analog inclusion $\text{Gen}_k(A) \subset \text{Lg}_k(A)$ does *not* hold. For a counterexample, consider $A = C([0, 1]^2, M_3)$. Then $\text{sr}(A) = 2$ and so $\text{Lg}_1(A)$ is not dense in A . On the other hand, $\text{Gen}_1(A)$ is dense in A since $\text{gr}(A) = 1$, see Theorem 4.23. This also shows that $\text{sr}(A) \not\leq \text{gr}'(A)$.

Proposition 2.5. *Let A be a C^* -algebra. Then $\text{rr}(A) \leq \text{gr}(A)$.*

Proof. This follows immediately from the the definition of real and generator rank together with the inclusion $\text{Gen}_k(\tilde{A})_{\text{sa}} \subset \text{Lg}_k(\tilde{A})_{\text{sa}}$ for every $k \geq 1$, which is shown in Remark 2.4. \square

While many C^* -algebras have real rank zero, the case of generator rank zero is very special:

Lemma 2.6. *Let A be a C^* -algebra. Then $\text{gr}(A) = 0$ if and only if A is a separable, commutative C^* -algebra with zero-dimensional spectrum.*

Proof. If $\text{gr}(A) = 0$, then A contains a generating self-adjoint element, and so A is separable and commutative. Thus, it remains to show that $\text{gr}(A) = 0$ if and only if $\dim(\text{Prim}(A)) = 0$ under the assumption that A is separable and commutative. Note that for every locally compact, second countable, Hausdorff space X , $\dim(X) = 0$ if and only if $\dim(X \times X) = 0$. Therefore, the result follows from Proposition 4.7. \square

It is easy to see that $\text{Lg}_k(A) \subset A^k$ and $\text{Lg}_k(A)_{\text{sa}} \subset A_{\text{sa}}^k$ are open subsets. For the sets of generating tuples, we have the following result:

Proposition 2.7. *Let A be a C^* -algebra, and let $k \in \mathbb{N}$. Then $\text{Gen}_k(A) \subset A^k$ and $\text{Gen}_k(A)_{\text{sa}} \subset A_{\text{sa}}^k$ are G_δ -subsets.*

Proof. We show that $\text{Gen}_k(A) \subset A^k$ is a G_δ -subset. The empty set is clearly G_δ , so we may assume $\text{Gen}_k(A) \neq \emptyset$, which in turn implies that A is separable. Let a_1, a_2, \dots be a dense sequence in A . Define:

$$U_n := \{\mathbf{x} \in A^k \mid a_1, \dots, a_n \in {}_{1/n}C^*(\mathbf{x})\}.$$

Let us check that $U_n \subset A^k$ is open. So let $\mathbf{x} \in U_n$. Then there exist polynomials p_1, \dots, p_n such that $a_i = {}_{1/n}p_i(\mathbf{x})$ for $i = 1, \dots, n$. We may consider each p_i as a function $A^k \rightarrow A$, which is clearly continuous. Therefore, for each i , there exists $\delta_i > 0$ such that $a_i = {}_{1/n}p_i(\mathbf{y})$ for all $\mathbf{y} \in A^k$ with $\mathbf{y} =_{\delta_i} \mathbf{x}$. Then the open ball around \mathbf{x} with radius $\min\{\delta_1, \dots, \delta_n\}$ is contained in U_n , which is therefore open.

One checks that $\text{Gen}_k(A) = \bigcap_{n \geq 1} U_n$, which completes the proof for $\text{Gen}_k(A)$. The result for $\text{Gen}_k(A)_{\text{sa}}$ is proved analogously. \square

Remark 2.8. Let A be a unital C^* -algebra. It is a consequence of Remark 2.4 and Proposition 2.7 that $\text{gr}(A) \leq 1$ if and only if the set of generators $\text{Gen}_1(A)$ forms a generic subset of A , i.e., if and only if $\text{Gen}_1(A) \subset A$ is a dense G_δ -set.

Our main tool to construct generators is the following Lemma 2.9, which was obtained together with Karen Strung, Aaron Tikuisis, Joav Orovitz and Stuart White at the workshop “Set theory and C^* -algebras” at the AIM in Palo Alto, January 2012.

It reduces the problem of showing $\text{gr}_{\text{sa}}(A) \leq k$. Instead of proving that every tuple can be approximated arbitrarily closely by tuples that generate the whole C^* -algebra, it is enough to show that every tuple can be approximated by tuples that approximately generate a single given element of A .

Lemma 2.9. *Let A be a separable C^* -algebra, let $k \in \mathbb{N}$, and let $S \subset A^k$ be a closed subset. Assume that for every $\mathbf{x} \in S$, every $\varepsilon > 0$, and every $z \in A$ there exists $\mathbf{y} \in S$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in {}_\varepsilon C^*(\mathbf{y})$. Then $\text{Gen}_k(A) \cap S \subset S$ is dense, i.e., for every $\mathbf{x} \in S$ and $\varepsilon > 0$ there exists $\mathbf{y} \in S$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $A = C^*(\mathbf{y})$.*

Proof. Let $a_2, a_3, \dots \in A$ be a dense sequence of A , where each element is repeated infinitely many times, and set $a_1 = 0$. We inductively find tuples $\mathbf{y}_k \in S$ and numbers $\delta_n > 0$ with the following properties:

- (1) $\|\mathbf{y}_n - \mathbf{y}_{n-1}\| < \min\{\delta_1/2^{n-1}, \delta_2/2^{n-2}, \dots, \delta_{n-1}/2\}$,
- (2) $a_n \in {}_{1/n}C^*(\mathbf{y}')$ whenever $\mathbf{y}' =_{\delta_n} \mathbf{y}_n$.

Set $\mathbf{y}_1 := \mathbf{x}$ and $\delta_1 := \varepsilon$. Then (2) is trivially satisfied.

Assume \mathbf{y}_i and δ_i have been constructed for $i \leq n-1$. By assumption, we can find $\mathbf{y}_n \in S$ satisfying (1) and such that $a_n \in {}_{1/n}C^*(\mathbf{y}_n)$. Then there exists a polynomial p such that $a_n = {}_{1/n}p(\mathbf{y}_n)$. We may consider p as a function $A^k \rightarrow A$, which is continuous. Therefore, there exists $\delta_n > 0$ satisfying (2).

Condition (1) ensures in particular that \mathbf{y}_n is a Cauchy sequence. Set $\mathbf{y} := \lim_n \mathbf{y}_n \in S$, and let us check that it has the desired properties.

For each n , repeated application of (1) gives:

$$\|\mathbf{y} - \mathbf{y}_n\| < \sum_{i \geq 1} \delta_n/2^i = \delta_n.$$

Thus, $\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{y} - \mathbf{y}_1\| < \delta_1 = \varepsilon$. Moreover, condition (2) ensures that $a_n \in {}_{1/n}C^*(\mathbf{y})$ for all k . It follows that $a_n \in C^*(\mathbf{y})$, since a_n was assumed to appear infinitely many times in the sequence a_1, a_2, \dots . Since the sequence a_n is dense in A , it follows $A = C^*(\mathbf{y})$, as desired. \square

2.10. When we want to estimate the generator rank of an ideal, we have to be careful about adjoining a unit. Mainly for technical reasons, we introduce the following variant of the generator rank:

$$(2.1) \quad \text{gr}_{\text{sa}}(A) \leq k \Leftrightarrow \text{Gen}_{k+1}(A)_{\text{sa}} \subset (A_{\text{sa}})^{k+1} \text{ is dense.}$$

By definition, $\text{gr}(A) = \text{gr}_{\text{sa}}(\tilde{A})$. The connection between the generator rank and its variant is summarized in the next result.

Lemma 2.11. *Let A be a C^* -algebra. Then:*

$$\text{gr}(A) = \max\{\text{rr}(A), \text{gr}_{\text{sa}}(A)\}.$$

Proof. If A is unital, the statement follows from Proposition 2.5. So assume A is non-unital, and denote by $1 \in \tilde{A}$ the adjoint unit. Let $\pi: \tilde{A} \rightarrow \mathbb{C}$ be the quotient morphism. It induces a natural morphism $\tilde{A}^k \rightarrow \mathbb{C}^k$, which we also denote by π . Let $\sigma: \mathbb{C} \rightarrow \tilde{A}$ denote the canonical split of π . We denote the induced morphism $\sigma: \mathbb{C}^k \rightarrow \tilde{A}^k$ also by σ .

Let us show $\text{gr}_{\text{sa}}(A) \leq \text{gr}_{\text{sa}}(\tilde{A}) = \text{gr}(A)$. Let $k := \text{gr}(A) + 1$, and we may assume this is finite. We want to verify the conditions of Lemma 2.9 for $S = A_{\text{sa}}^k$. So let $\mathbf{x} \in A_{\text{sa}}^k$, $\varepsilon > 0$ and $z \in A$ be given. By assumption, there exists $\mathbf{y} \in \tilde{A}_{\text{sa}}^k$ with $\mathbf{x} =_{\varepsilon/2} \mathbf{y}$ and a polynomial q such that $z =_{\varepsilon/2} q(\mathbf{y})$.

There is a unique decomposition $\mathbf{y} = \mathbf{a} + \mathbf{r}$ for $\mathbf{r} = \sigma \circ \pi(\mathbf{x}) \in (\mathbb{C}1)^k$ and $\mathbf{a} \in A_{\text{sa}}^k$. Then $\|\mathbf{r}\| < \varepsilon/2$, and therefore $\mathbf{x} =_{\varepsilon/2} \mathbf{y} =_{\varepsilon/2} \mathbf{a}$. Note that $p(\mathbf{y}) = p(\mathbf{a} + \mathbf{r})$ has the form $q(\mathbf{a}) + \lambda 1$ for some polynomial q and a constant $\lambda \in \mathbb{C}$. Since $q(\mathbf{a}) \in A^k$ and $p(\mathbf{y}) =_{\varepsilon/2} z \in A^n$, we get $|\lambda| \leq \varepsilon/2$. Then $q(\mathbf{a}) =_{\varepsilon} z$, and so $z \in_{\varepsilon} C^*(\mathbf{a})$, which shows that \mathbf{a} has the desired properties. It follows from Lemma 2.9 that $\text{Gen}_k(A)_{\text{sa}} \subset A_{\text{sa}}^k$ is dense, and so $\text{gr}_{\text{sa}}(A) \leq \text{gr}(A)$, as desired.

It was shown in Proposition 2.5 that $\text{rr}(A) \leq \text{gr}(A)$. Thus, it remains to show $\text{gr}(A) \leq \max\{\text{rr}(A), \text{gr}_{\text{sa}}(A)\}$.

Let $k := \max\{\text{rr}(A), \text{gr}_{\text{sa}}(A)\} + 1$, and we may assume this is finite. Let $\mathbf{x} \in \tilde{A}_{\text{sa}}^k$ and $\varepsilon > 0$ be given. By assumption, there is $\mathbf{x}' \in \text{Lg}_k(\tilde{A})_{\text{sa}}$ with $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$. Since $\text{Lg}_k(\tilde{A})_{\text{sa}}$ is open, there exists $\delta > 0$ such that $\mathbf{b} \in \text{Lg}_k(\tilde{A})_{\text{sa}}$ whenever $\mathbf{b} \in \tilde{A}_{\text{sa}}^k$ satisfies $\mathbf{b} =_{\delta} \mathbf{x}'$. We may assume $\delta < \varepsilon/2$.

There is a unique decomposition $\mathbf{x}' = \mathbf{a} + \mathbf{r}$ for $\mathbf{r} = \sigma \circ \pi(\mathbf{x}) \in (\mathbb{C}1)^k$ and $\mathbf{a} \in A_{\text{sa}}^k$. By assumption, there exists $\mathbf{a}' \in A_{\text{sa}}^k$ with $\mathbf{a}' =_{\delta} \mathbf{a}$ and such that $C^*(\mathbf{a}') = A$. Set $\mathbf{y} := \mathbf{a}' + \mathbf{r}$. Note that $\mathbf{y} =_{\delta} \mathbf{x}'$, and therefore $\mathbf{y} \in \text{Lg}_k(\tilde{A})_{\text{sa}}$. It follows $1 \in C^*(\mathbf{y})$, and so $a'_i = y_i - r_i 1 \in C^*(\mathbf{y})$ for $i = 1, \dots, k$. Thus, \mathbf{y} generates \tilde{A} . Moreover, $\mathbf{y} =_{\varepsilon} \mathbf{x}$. We have shown that $\text{Gen}_k(\tilde{A})_{\text{sa}}$ is dense in \tilde{A}_{sa}^k , and so $\text{gr}(A) \leq k - 1$, as desired. \square

Proposition 2.12. *Let A be a C^* -algebra, and let $J \triangleleft A$ be an ideal. Then $\text{gr}(A/J) \leq \text{gr}(A)$.*

Proof. Note that J is also an ideal in \tilde{A} , and $\tilde{A}/J \cong \widetilde{A/J}$. Let $\pi: \tilde{A} \rightarrow \tilde{A}/J$ denote the quotient morphism. It induces a surjective morphism $\tilde{A}_{\text{sa}}^k \rightarrow \widetilde{A/J}_{\text{sa}}^k$, which sends $\text{Gen}_k(\tilde{A})_{\text{sa}}$ into $\text{Gen}_k(\tilde{A}/J)_{\text{sa}}$. Thus, if $\text{Gen}_k(\tilde{A})_{\text{sa}} \subset \tilde{A}_{\text{sa}}^k$ is dense, then so is $\text{Gen}_k(\tilde{A}/J)_{\text{sa}} \subset \widetilde{A/J}_{\text{sa}}^k$. This shows $\text{gr}(A/J) \leq \text{gr}(A)$. \square

One immediate consequence of the key lemma Lemma 2.9 is that the generator rank behaves well with respect to approximation by sub- C^* -algebras and inductive limits.

Recall that a collection $A_i \subset A$ of sub- C^* -algebras is said to **approximate** A if for every finite subset $F \subset A$ and for every $\varepsilon > 0$, there exists i such that $F \subset_\varepsilon A_i$.

Proposition 2.13. *Let A be a separable C^* -algebra, and let $k \geq 0$. Assume A is approximated by sub- C^* -algebras $A_i \subset A$ with $\text{gr}(A_i) \leq k$. Then $\text{gr}(A) \leq k$.*

Moreover, if $A = \varinjlim A_n$ is an inductive limit, then $\text{gr}(A) \leq \liminf_n \text{gr}(A_n)$.

Proof. Assume A is approximated by a collection $A_i \subset A$ with $\text{gr}(A_i) \leq k$. For each i , we have $\text{rr}(A_i) \leq k$ by Proposition 2.5. It follows $\text{rr}(A) \leq k$, since the real rank behaves well with respect to approximation by subalgebras, as noted in [Thi12, Remark 2]. Thus, by Lemma 2.11, it is enough to show $\text{gr}_{\text{sa}}(A) \leq k$.

We want to verify the conditions of Lemma 2.9 for $S = A_{\text{sa}}^{k+1}$. So let $\mathbf{x} \in A_{\text{sa}}^{k+1}$, $\varepsilon > 0$ and $z \in A$ be given. Since the A_i approximate A , there exists an index i such that there is $\mathbf{x}' \in (A_i)_{\text{sa}}^{k+1}$ with $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$, and such that there is $z' \in A_i$ with $z' =_\varepsilon z$. Since $\text{gr}(A_i) \leq k$, there exists $\mathbf{y} \in \text{Gen}_{k+1}(A_i)_{\text{sa}}$ with $\mathbf{y} =_{\varepsilon/2} \mathbf{x}'$. Then $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in_\varepsilon C^*(\mathbf{y})$, as desired. It follows from Lemma 2.9 that $\text{gr}_{\text{sa}}(A) \leq k$.

This result, together with Proposition 2.12, implies the estimate for an inductive limit. The argument is standard for dimension theories, see [Thi12, Proposition 2], but for the convenience of the reader we include a short proof.

Assume $A = \varinjlim A_n$. For each n , let B_n be the image of A_n in the inductive limit A . Then B_n is a quotient of A_n , and therefore $\text{gr}(B_n) \leq \text{gr}(A_n)$, by Proposition 2.12. Note that A is approximated by the collection $(B_n)_{n \in J}$ whenever $J \subset \mathbb{N}$ is cofinal. In that case, it follows from the above result that $\text{gr}(A) \leq \sup_{n \in J} \text{gr}(B_n)$. Since this holds for every cofinal subset $J \subset \mathbb{N}$, we obtain:

$$\text{gr}(A) \leq \inf_{n \in J} \{\sup \text{gr}(B_n) \mid J \subset \mathbb{N} \text{ cofinal}\} = \liminf_n \text{gr}(B_n) \leq \liminf_n \text{gr}(A_n),$$

as desired. \square

Theorem 2.14. *Let A be a C^* -algebra, and let $J \triangleleft A$ be an ideal. Then $\text{gr}(J) \leq \text{gr}(A)$.*

Proof. Note that J is also an ideal in \tilde{A} and $\text{gr}(\tilde{A}) = \text{gr}(A)$. Thus, we may assume from now on that A is unital,

It is known that the real rank behaves well with respect to ideals, i.e., $\text{rr}(J) \leq \text{rr}(A)$, see [EH95, Théorème 1.4]. We have $\text{rr}(A) \leq \text{gr}(A)$ by Proposition 2.5. Thus, by Lemma 2.11, it remains to show $\text{gr}_{\text{sa}}(J) \leq \text{gr}(A)$. Let $k := \text{gr}(A) + 1$, and we may assume this is finite. Then A and J are separable, and so there exists a sequential, quasi-central approximate unit $(h_\alpha) \subset J_+$, see [AP77, Corollary 3.3] and [Arv77]. We may assume $\|h_\alpha\| \leq 1$.

For a vector $\mathbf{a} = (a_1, \dots, a_k) \in A^k$, we will use the following notation:

$$|\mathbf{a}| := \sum_i |a_i| = \sum_i (a_i^* a_i)^{1/2},$$

$$\mathbf{a}^{(\alpha)} := (h_\alpha^{1/2} a_1 h_\alpha^{1/2}, \dots, h_\alpha^{1/2} a_k h_\alpha^{1/2}).$$

For $a \in A_+$, we denote by $\text{Her}(a) := aAa$ the hereditary sub- C^* -algebra generated by a . We will consider the sequence algebra $Q := \prod_\alpha A / \bigoplus_\alpha A$. For an element $s \in A$, we denote by $\langle s \rangle \in Q$ the image of the constant sequence. We denote by $\langle h_\alpha \rangle \in Q$ the image of the sequence (h_α) . Note that $\langle h_\alpha \rangle$ commutes with $\langle s \rangle$ in Q .

To show $\text{gr}_{\text{sa}}(J) \leq k - 1$, we want to verify the conditions of Lemma 2.9. So let $\mathbf{x} \in J_{\text{sa}}^k$, $\varepsilon > 0$ and $z \in J$ be given. In 5 steps, we will construct $\mathbf{y} \in J_{\text{sa}}^k$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in_\varepsilon C^*(\mathbf{y})$.

Step 1: We will find $\mathbf{x}' \in J_{\text{sa}}^k$ and $\delta > 0$ such that $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$ and $z \in_{\varepsilon/4} \text{Her}((|\mathbf{x}'| - \delta)_+)$.

By assumption, there exists $\mathbf{a} \in \text{Gen}_k(A)_{\text{sa}}$ with $\mathbf{a} =_{\varepsilon/4} \mathbf{x}$. Then $|\mathbf{a}|$ is invertible in A , and so $|\mathbf{a}| \geq 3\delta$ for some $\delta > 0$. Choose an index α_0 large enough such that for all $\alpha \geq \alpha_0$ the following conditions hold:

$$(2.2) \quad z \ll_{\varepsilon/8} \frac{1}{\delta} (3\delta h_\alpha - 2\delta)_+,$$

$$(2.3) \quad \mathbf{x}^{(\alpha)} =_{\varepsilon/4} \mathbf{x},$$

$$(2.4) \quad |\mathbf{a}^{(\alpha)}| =_\delta h_\alpha^{1/2} |\mathbf{a}| h_\alpha^{1/2}.$$

Then, using (2.4) at the second step, we get:

$$3\delta h_\alpha - 2\delta \leq h_\alpha^{1/2} |\mathbf{a}| h_\alpha^{1/2} - 2\delta \leq |\mathbf{a}^{(\alpha)}| - \delta.$$

In general, if two commuting, self-adjoint elements c, d satisfy $c \leq d$, then $c_+ \leq d_+$, but this does not necessarily hold if c and d do not commute. Thus, we may not deduce $(3\delta h_\alpha - 2\delta)_+ \leq (|\mathbf{a}^{(\alpha)}| - \delta)_+$. However, using the sequence algebra Q , we will show that this holds up to an arbitrarily small tolerance for sufficiently large α . Indeed, the elements $\langle 3\delta h_\alpha - 2\delta \rangle$ and $\langle |\mathbf{a}^{(\alpha)}| - \delta \rangle$ commute, and therefore:

$$\langle (3\delta h_\alpha - 2\delta)_+ \rangle \leq \langle (|\mathbf{a}^{(\alpha)}| - \delta)_+ \rangle.$$

Therefore, for $\alpha \geq \alpha_0$ large enough we have:

$$(2.5) \quad \frac{1}{\delta} (3\delta h_\alpha - 2\delta)_+ \in_{\varepsilon/8} \text{Her}((|\mathbf{a}^{(\alpha)}| - \delta)_+).$$

For such α , we set $\mathbf{x}' := \mathbf{a}^{(\alpha)}$. Let us verify that \mathbf{x}' has the desired properties. From $\mathbf{a} =_{\varepsilon/4} \mathbf{x}$ we get $\mathbf{a}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}^{(\alpha)}$, and we deduce, using (2.3) at the third step:

$$\mathbf{x}' = \mathbf{a}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}.$$

Moreover, it follows from (2.2) and (2.5) that $z \in_{\varepsilon/4} \text{Her}((|\mathbf{x}'| - \delta)_+)$, as desired.

Step 2: Since $z \in_{\varepsilon/4} \text{Her}((|\mathbf{x}'| - \delta)_+)$, there exists a polynomial p such that:

$$z \ll_{\varepsilon/4} p(\mathbf{x}') \cdot (|\mathbf{x}'| - \delta)_+.$$

Set $M = 2 \cdot \max\{\|p(\mathbf{x}')\|, \|(|\mathbf{x}'| - \delta)_+\|\}$. Let $\eta > 0$ be such that for all $\mathbf{b} \in A_{\text{sa}}^k$ with $\mathbf{b} =_\eta \mathbf{x}'$ we have:

$$(2.6) \quad z \ll_{\varepsilon/4} p(\mathbf{b}) \cdot (|\mathbf{b}| - \delta)_+,$$

$$(2.7) \quad |\mathbf{b}| =_\delta |\mathbf{x}'|,$$

$$(2.8) \quad \|p(\mathbf{b})\| \leq M,$$

$$(2.9) \quad \|(|\mathbf{b}| - \delta)_+\| \leq M.$$

We may assume $\eta < \varepsilon/2$ and $\eta < \delta$.

Step 3: Since $\text{gr}(A) \leq k - 1$, there exists $\mathbf{c} \in A_{\text{sa}}^k$ with $\mathbf{c} =_\eta \mathbf{x}'$ and a polynomial q such that

$$(2.10) \quad z =_{\varepsilon/(4M^2)} q(\mathbf{c}).$$

Then q can be decomposed as a finite sum of polynomials, $q = \sum_{d=1}^N q_d$, where q_d is homogeneous of degree d , i.e., $q_d(t\mathbf{c}) = t^d q_d(\mathbf{c})$ for every $t \in \mathbb{R}_+$. Set

$$(2.11) \quad L := \max\{\|q_1(\mathbf{c})\|, \dots, \|q_N(\mathbf{c})\|\}.$$

Step 4: We show that for every tolerance $\sigma > 0$ we have $(|\mathbf{c}^{(\alpha)}| - \delta)_+ \ll_\sigma h_\alpha$ for all α large enough.

To that end, let us verify that $\lim_{\alpha \rightarrow \infty} \|(|\mathbf{c}^{(\alpha)}| - \delta)_+ \cdot (1 - h_\alpha)\| = 0$, which means:

$$\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \rangle \cdot \langle 1 - h_\alpha \rangle = 0,$$

in Q .

We have $\mathbf{c} =_\eta \mathbf{x}'$, and therefore $|\mathbf{y}| =_\delta |\mathbf{x}'|$ by (2.7). Let $\pi: A \rightarrow B$ denote the quotient morphism. Since $\pi(|\mathbf{x}'|) = 0$, we get $\pi(|\mathbf{c}| - \delta) \leq 0$ and so $\pi((|\mathbf{c}| - \delta)_+) = 0$. Thus, $\langle (|\mathbf{c}| - \delta)_+ \rangle \cdot \langle 1 - h_\alpha \rangle = 0$, in Q .

For $i = 1, \dots, k$ we have $\langle |h_\alpha^{1/2} c_i h_\alpha^{1/2}| \rangle = \langle |c_i|^{1/2} h_\alpha |c_i|^{1/2} \rangle \leq \langle c_i \rangle$. It follows that $\langle |\mathbf{c}^{(\alpha)}| \rangle \leq \langle |\mathbf{c}| \rangle$, and since these two elements commute, we get $\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \rangle \leq \langle (|\mathbf{c}| - \delta)_+ \rangle$. It follows $\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \rangle \langle 1 - h_\alpha \rangle = 0$, as desired.

Step 5: We choose the index α large enough satisfying (2.12), (2.13) and (2.14) below for $\mathbf{y} := \mathbf{c}^{(\alpha)}$. For (2.12), this is possible since $\mathbf{c} =_\eta \mathbf{x}' \in J_{\text{sa}}^k$ and so $\mathbf{c}^{(\alpha)} =_\eta (\mathbf{x}')^{(\alpha)}$ for all α , and $\lim_{\alpha \rightarrow \infty} (\mathbf{x}')^{(\alpha)} = \mathbf{x}'$. For (2.13), this is possible by Step 4. Finally, for (2.14) this is possible since (h_α) is quasi-central.

$$(2.12) \quad \mathbf{y} =_\eta \mathbf{x}'$$

$$(2.13) \quad (|\mathbf{y}| - \delta)_+ \ll_{\varepsilon/(4MNL)} h_\alpha^d, \text{ for } d = 1, \dots, N.$$

$$(2.14) \quad q(\mathbf{y}) = \sum_{d=1}^N q_d(\mathbf{y}) =_{\varepsilon/(4M^2)} \sum_{d=1}^N h_\alpha^d \cdot q_d(\mathbf{c}).$$

Let us check that $\mathbf{y} = \mathbf{c}^{(\alpha)}$ has the desired properties.

Since $\eta \leq \varepsilon/2$, we get, using (2.12) for the first estimate, and using Step 1 for the second estimate:

$$\mathbf{y} =_{\varepsilon/2} \mathbf{x}' =_{\varepsilon/2} \mathbf{x}.$$

It remains to check $z \in_\varepsilon C^*(\mathbf{y})$. Since $\mathbf{y} =_\eta \mathbf{x}'$, we get from Step 2 that the estimates (2.6)-(2.9) hold for \mathbf{y} . We compute, using (2.14), (2.8) and (2.9) at the first step, using (2.13), (2.8) and (2.11) at the third step, (2.10) at the fifth step, and (2.6) at the last step:

$$\begin{aligned} p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot q(\mathbf{y}) &=_{\varepsilon/4} p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot \sum_{d=1}^N h_\alpha^d \cdot q_d(\mathbf{c}) \\ &= p(\mathbf{y}) \cdot \sum_{d=1}^N (|\mathbf{y}| - \delta)_+ \cdot h_\alpha^d \cdot q_d(\mathbf{c}) \\ &=_{\varepsilon/4} p(\mathbf{y}) \cdot \sum_{d=1}^N (|\mathbf{y}| - \delta)_+ \cdot q_d(\mathbf{c}) \\ &= p(\mathbf{y}) (|\mathbf{y}| - \delta)_+ \cdot \sum_{d=1}^N q_d(\mathbf{y}) \\ &=_{\varepsilon/4} p(\mathbf{y}) (|\mathbf{y}| - \delta)_+ \cdot z \\ &=_{\varepsilon/4} z \end{aligned}$$

Since $p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot q(\mathbf{y}) \in C^*(\mathbf{y})$, we have verified $z \in_\varepsilon C^*(\mathbf{y})$, as desired.

By Lemma 2.9, it follows $\text{gr}_{\text{sa}}(J) \leq k - 1$, as desired. \square

In [Thi12, Definition 1], the concept of a non-commutative dimension theory was formalized by proposing a set of axioms. These axioms are generalizations of properties of the dimension of locally compact, Hausdorff spaces, and it was shown that they are satisfied by many theories, in particular the real and stable rank, the topological dimension, the decomposition rank and the nuclear dimension.

Definition 2.15 ([Thi12, Definition 1]). Let \mathcal{C} be a class of C^* -algebras that is closed under $*$ -isomorphisms, and closed under taking ideals, quotients, finite direct sums, and minimal unitizations. A **dimension theory** for \mathcal{C} is an assignment $d: \mathcal{C} \rightarrow \overline{\mathbb{N}} = \{0, 1, 2, \dots, \infty\}$ such that $d(A) = d(A')$ whenever A, A' are isomorphic C^* -algebras in \mathcal{C} , and moreover the following axioms are satisfied:

- (D1) $d(J) \leq d(A)$ whenever $J \triangleleft A$ is an ideal in $A \in \mathcal{C}$,
- (D2) $d(A/J) \leq d(A)$ whenever $J \triangleleft A \in \mathcal{C}$,
- (D3) $d(A \oplus B) = \max\{d(A), d(B)\}$, whenever $A, B \in \mathcal{C}$,
- (D4) $d(\tilde{A}) = d(A)$, whenever $A \in \mathcal{C}$.
- (D5) If $A \in \mathcal{C}$ is approximated by subalgebras $A_i \in \mathcal{C}$ with $d(A_i) \leq n$, then $d(A) \leq n$.
- (D6) Given $A \in \mathcal{C}$ and a separable sub- C^* -algebra $C \subset A$, there exists a separable C^* -algebra $D \in \mathcal{C}$ such that $C \subset D \subset A$ and $d(D) \leq d(A)$.

Note that we do not assume that \mathcal{C} is closed under approximation by sub- C^* -algebra, so that the assumption $A \in \mathcal{C}$ in (D5) is necessary. Moreover, in axiom (D6), we do not assume that the separable subalgebra C lies in \mathcal{C} .

Remark 2.16. Let us consider the generator rank on the class of separable C^* -algebras.

We have verified axioms (D1) in Theorem 2.14, (D2) in Proposition 2.12 and (D5) in Proposition 2.13. Note that (D4) holds by definition, and (D6) is superfluous if we only consider separable C^* -algebras.

The question remains whether axiom (D3) holds, that is, whether $\text{gr}(A \oplus B) = \max\{\text{gr}(A), \text{gr}(B)\}$, and this turns out to be surprisingly difficult. We can only verify it in specific cases, namely for C^* -algebras of real rank zero, see Proposition 3.1, or for homogeneous C^* -algebras, see Corollary 4.27. We conjecture that (D3) for the generator rank holds in general.

The next result gives an estimate of the generator rank of an extension of C^* -algebras in terms of the generator rank of ideal and quotient. We remark that no such estimate is known for the real rank.

Theorem 2.17. *Let A be a C^* -algebra, and let $J \triangleleft A$ be an ideal. Then:*

$$\text{gr}(A) \leq \text{gr}(J) + \text{gr}(A/J) + 1.$$

Proof. Since J is also an ideal in \tilde{A} , and $\text{gr}(A) = \text{gr}(\tilde{A})$, we may assume that A is unital.

Set $B := A/J$. Let $k := \text{gr}(J) + 1$ and $l := \text{gr}(B) + 1$, which we may assume are finite. Let $\pi: A \rightarrow B$ denote quotient morphism. It induces a natural morphism $A^l \rightarrow B^l$, which we also denote by π . Given $\mathbf{x} \in A_{\text{sa}}^k$, $\mathbf{y} \in A_{\text{sa}}^l$ and $\varepsilon > 0$, we want to find $\mathbf{x}' \in A_{\text{sa}}^k$, $\mathbf{y}' \in A_{\text{sa}}^l$ such that $\mathbf{x}' =_\varepsilon \mathbf{x}$, $\mathbf{y}' =_\varepsilon \mathbf{y}$ and $A = C^*(\mathbf{x}', \mathbf{y}')$.

Let $\mathbf{b} := \pi(\mathbf{y})$. Since $\text{gr}(B) \leq l - 1$, we may find $\mathbf{b}' \in \text{Gen}_l(B)_{\text{sa}}$ with $\mathbf{b}' =_\varepsilon \mathbf{b}$. Let $\mathbf{y}' \in A_{\text{sa}}^l$ be a lift of \mathbf{b}' with $\mathbf{y}' =_\varepsilon \mathbf{y}$. For $i = 1, \dots, k$, choose an element $a_i \in C^*(\mathbf{y}')$ such that $\pi(a_i) = \pi(x_i)$. Set $\mathbf{a} = (a_1, \dots, a_k) \in A_{\text{sa}}^k$. Note that $\mathbf{x} - \mathbf{a} \in J_{\text{sa}}^k$. Since $\text{gr}_{\text{sa}}(J) \leq \text{gr}(J) \leq k - 1$, we may find $\mathbf{c} \in \text{Gen}_k(J)_{\text{sa}}$ with $\mathbf{c} =_\varepsilon \mathbf{x} - \mathbf{a}$. Set $\mathbf{x}' := \mathbf{a} + \mathbf{c}$. Then \mathbf{x}' and \mathbf{y}' have the desired properties. \square

3. THE GENERATOR RANK OF REAL RANK ZERO C^* -ALGEBRAS

In this section, we restrict our attention to separable C^* -algebras with real rank zero. On this class of C^* -algebras, the generator rank is a dimension theory in the sense of Definition 2.15, see Remark 2.16 and Proposition 3.1.

We then show that AF-algebras have generator rank at most one, see Corollary 3.3.

In Theorem 3.6, we prove the estimate $\text{gr}(A \otimes M_n) \leq \left\lceil \frac{\text{gr}(A)}{n^2} \right\rceil$ under the additional assumption that A is unital and has stable rank. This shows that for such algebras the generator rank decreases when tensoring with matrix algebras of higher and higher dimension. Thus, if A is a separable, unital, real rank zero, stable rank one C^* -algebra with finite generator rank, then $\text{gr}(A \otimes B) = 1$ for any infinite UHF-algebra B . We generalize this by showing that every separable, real rank zero C^* -algebra that tensorially absorbs a UHF-algebra has generator rank at most one, see Proposition 3.8.

Proposition 3.1. *Let A, B be C^* -algebras of real rank zero. Then $\text{gr}(A \oplus B) = \max\{\text{gr}(A), \text{gr}(B)\}$.*

Proof. We have $\text{gr}(A), \text{gr}(B) \leq \text{gr}(A \oplus B)$ by Proposition 2.12 (or Theorem 2.14), and it therefore remains to show $\text{gr}(A \oplus B) \leq \max\{\text{gr}(A), \text{gr}(B)\}$. Since $A \oplus B$ is an ideal in $\tilde{A} \oplus \tilde{B}$, we obtain $\text{gr}(A \oplus B) \leq \text{gr}(\tilde{A} \oplus \tilde{B})$ from Theorem 2.14. We have $\text{gr}(A) = \text{gr}(\tilde{A})$ and $\text{gr}(B) = \text{gr}(\tilde{B})$ by definition, and thus it remains to show $\text{gr}(\tilde{A} \oplus \tilde{B}) \leq \max\{\text{gr}(\tilde{A}), \text{gr}(\tilde{B})\}$. We may therefore assume that A and B are unital.

So let A, B be unital, real rank zero C^* -algebras. Let $k := \max\{\text{gr}(A), \text{gr}(B)\} + 1$, and we may assume this is finite. We want to verify the conditions of Lemma 2.9. Let $\mathbf{a} \in A_{\text{sa}}^k$, $\mathbf{b} \in B_{\text{sa}}^k$, $\varepsilon > 0$ and $x \in A$, $y \in B$ be given. We need to find $\mathbf{c} \in A_{\text{sa}}^k$, $\mathbf{d} \in B_{\text{sa}}^k$ such that $\mathbf{c} =_{\varepsilon} \mathbf{a}$, $\mathbf{d} =_{\varepsilon} \mathbf{b}$ and $(x \oplus y) \in_{\varepsilon} C^*(\mathbf{c} \oplus \mathbf{d})$, where we use the notation $\mathbf{a}' \oplus \mathbf{b}' = (a'_1 \oplus b'_1, \dots, a'_k \oplus b'_k) \in (A \oplus B)^k$ for the direct sum of tuples.

Since $\text{rr}(A) = \text{rr}(B) = 0$, we may perturb a_1 and b_1 to be invertible, self-adjoint and have disjoint (finite) spectra. More precisely, there are $a'_1 \in A_{\text{sa}}$ and $b'_1 \in B_{\text{sa}}$ such that $a'_1 =_{\varepsilon/2} a_1$, $b'_1 =_{\varepsilon/2} b_1$ and $\sigma(a'_1) \cap \sigma(b'_1) = \emptyset$ and $0 \notin \sigma(a'_1), 0 \notin \sigma(b'_1)$. Let $\delta_0 > 0$ be smaller than the distance between any two points in $\sigma(a'_1) \cup \sigma(b'_1) \cup \{0\}$. Define continuous functions $f, g: \mathbb{R} \rightarrow [0, 1]$ such that:

- (1) f has value 1 on a $\delta_0/4$ -neighborhood of $\sigma(a'_1)$, and has value 0 on a $\delta_0/4$ -neighborhood of $\sigma(b'_1) \cup \{0\}$.
- (2) g has value 1 on a $\delta_0/4$ -neighborhood of $\sigma(b'_1)$, and has value 0 on a $\delta_0/4$ -neighborhood of $\sigma(a'_1) \cup \{0\}$.

Let $\delta > 0$ be such that:

- (1) Whenever $c_1 \in A_{\text{sa}}$ satisfies $c_1 =_{\delta} a'_1$, then the spectrum $\sigma(c_1)$ is contained in a $\delta_0/4$ -neighborhood of $\sigma(a'_1)$.
- (2) Whenever $d_1 \in B_{\text{sa}}$ satisfies $d_1 =_{\delta} b'_1$, then the spectrum $\sigma(d_1)$ is contained in a $\delta_0/4$ -neighborhood of $\sigma(b'_1)$.

We may assume $\delta < \varepsilon/2$.

By assumption, there exists $\mathbf{c} \in \text{Gen}_k(A)_{\text{sa}}$ with $\mathbf{c} =_{\delta} (a'_1, a_2, \dots, a_k)$, and there exists $\mathbf{d} \in \text{Gen}_k(B)_{\text{sa}}$ with $\mathbf{d} =_{\delta} (b'_1, b_2, \dots, b_k)$. Then $\mathbf{c} \oplus \mathbf{d} =_{\varepsilon} \mathbf{a} \oplus \mathbf{b}$, and we claim that $\mathbf{c} \oplus \mathbf{d}$ generates $A \oplus B$.

So let $x \oplus y \in A \oplus B$, and $\eta > 0$ be given. Since \mathbf{c} generates A , there exists a polynomial p such that $x =_{\eta} p(\mathbf{c})$. Similarly, there exists a polynomial q such that $y =_{\eta} q(\mathbf{d})$. By construction, $f(c_1) = 1_A$, $f(d_1) = 0_B$, $g(c_1) = 0_A$ and $g(d_1) = 1_B$. It

follows:

$$p(\mathbf{c}) \oplus q(\mathbf{d}) = f(c_1 \oplus d_1)p(\mathbf{c} \oplus \mathbf{d}) + g(c_1 \oplus d_1)q(\mathbf{c} \oplus \mathbf{d}) \in C^*(\mathbf{c} \oplus \mathbf{d}).$$

Since $x \oplus y \in A \oplus B$, and $\eta > 0$ were arbitrary, this shows $\mathbf{c} \oplus \mathbf{d} \in \text{Gen}_k(A \oplus B)_{\text{sa}}$.

Thus, $\text{Gen}_k(A \oplus B)_{\text{sa}}$ is dense in $(A \oplus B)_{\text{sa}}^k$, and so $\text{gr}(A \oplus B) \leq k - 1$, as desired. \square

Lemma 3.2. *For $n \geq 2$, we have $\text{gr}(M_n) = 1$.*

Proof. Let $a, b \in (M_n)_{\text{sa}}$ and $\varepsilon > 0$ be given. Then there exists a unitary such that uau^* is diagonal. Let $a' =_\varepsilon uau^*$ be self-adjoint and diagonal such that the spectrum $\sigma(a')$ contains n different non-zero entries. Let $b' =_\varepsilon ubu^*$ be a self-adjoint element such that the off-diagonal entries $b'_{i,i+1}$ are non-zero for $i = 1, \dots, n-1$. It is easily checked that $C^*(a', b') = M_n$.

Since conjugation by u is isometric, we have $u^*a'u =_\varepsilon a$ and $u^*b'u =_\varepsilon b$. Moreover, $C^*(u^*a'u, u^*b'u) = C^*(a', b') = M_n$. Thus, we have approximated the pair (a, b) by a generating pair, and since $\varepsilon > 0$ was arbitrary, we get that $\text{Gen}_2(M_n)_{\text{sa}} \subset (M_n)_{\text{sa}}^2$ is dense, as desired. \square

Corollary 3.3. *Let A be a separable AF-algebra. Then $\text{gr}(A) \leq 1$.*

Proof. It follows from Lemma 3.2 and Proposition 3.1 that $\text{gr}(B) \leq 1$ for every finite-dimensional C^* -algebra B . Then, the result for AF-algebras follows directly from Proposition 2.13. \square

We now turn towards the problem of estimating the generator rank of $A \otimes M_n$. An important ingredient is the fact that we can approximately diagonalize matrices, which is conceptualized by the following notion of Xue:

Definition 3.4 (Xue, [Xue10, Definition 3.1]). Let A be a C^* -algebra, and $n \geq 2$. An element $a \in A \otimes M_n$ is said to be **approximately diagonalizable** if for any $\varepsilon > 0$ there exist a unitary $u \in A \otimes M_n$ and $d_1, \dots, d_n \in A$ such that $\|uau^* - \text{diag}(d_1, \dots, d_n)\| < \varepsilon$.

We call A **approximately diagonal**, if for any $n \geq 2$, every self-adjoint element in $A \otimes M_n$ is approximately diagonalizable.

Proposition 3.5 (Zhang, [Zha90, Corollary 3.6]). *Every real rank zero C^* -algebra is approximately diagonal.*

Theorem 3.6. *Let A be a unital C^* -algebra with real rank zero and stable rank one, and $n \in \mathbb{N}$. If $\text{gr}(A) \geq 1$, then:*

$$\text{gr}(A \otimes M_n) \leq \left\lceil \frac{\text{gr}(A)}{n^2} \right\rceil.$$

In particular, if $\text{gr}(A) \leq n^2$, then $\text{gr}(A \otimes M_n) \leq 1$.

Proof. The inequality clearly holds if $n = 1$, so we may assume $n \geq 2$. Then also $\text{gr}(A \otimes M_n) \geq 1$.

Assume $\text{gr}(A) \leq (d-1) \cdot n^2$ for some $d \geq 2$. We need to show $\text{gr}(A \otimes M_n) \leq d-1$. So let $c^{(k)} = (c_{i,j}^{(k)})$, $k = 1, \dots, d$ be d self-adjoint matrices in $A \otimes M_n$. In several steps, we will show how to approximate these matrices by matrices $\tilde{c}^{(k)}$ that generate $A \otimes M_n$.

We let $e_{ij} \in M_n$ denote the canonical matrix units. To simplify notation, we set $a := c^{(1)}$ and $b := c^{(2)}$. In step 1, we will show that we may assume a is diagonal. In step 2, we will show that we may also assume that the entries $b_{1,n}, \dots, b_{n-1,n}$ are

positive, and in step 3 we show that we may further assume that the diagonal entries of a are invertible and have finite, disjoint spectra.

In step 1 and 2, we will find a unitary $u \in A \otimes M_n$ and consider the conjugated elements $uc^{(k)}u^*$. Note that it is enough to find generators close to these new elements $uc^{(k)}u^*$, since conjugation by u is isometric, and the elements $c^{(1)}, \dots, c^{(d)}$ generate $A \otimes M_n$ if and only if $uc^{(1)}u^*, \dots, uc^{(d)}u^*$ do.

Step 1: We show that we may assume $a = c^{(1)}$ is diagonal.

By Proposition 3.5, we may approximately diagonalize a , i.e., for every tolerance $\varepsilon > 0$ there exists a unitary $u \in A \otimes M_n$ and $d_1, \dots, d_n \in A$ such that $\|uau^* - \text{diag}(d_1, \dots, d_n)\| < \varepsilon$. As explained above, it is enough to find a generating tuple close to the conjugated matrices $uau^*, uc^{(2)}u^*, \dots, uc^{(d)}u^*$. Thus, by considering the conjugated elements, we may from now on assume that a is diagonal.

Step 2: We show that we may assume the $n - 1$ off-diagonal entries $b_{1,n}, \dots, b_{n-1,n}$ of $b = c^{(2)}$ are positive and invertible.

Since $\text{sr}(A) = 1$, for each $b_{1,n}, \dots, b_{n-1,n}$ we may find an invertible element in A that is arbitrarily close. Thus, by perturbing b , we may assume each $b_{i,n}$ is invertible in A .

For $i = 1, \dots, n - 1$, set $u_i := (b_{i,n}^* b_{i,n})^{-1/2} b_{i,n}^*$, which is a unitary in A . Note that $|b_{i,n}| := (b_{i,n}^* b_{i,n})^{1/2}$ is a positive, invertible element, and $|b_{i,n}| = u_i b_{i,n}$.

Set $u_n := 1$ and define a diagonal, unitary matrix as $u := \text{diag}(u_1, \dots, u_n) \in A \otimes M_n$. As explained above, it is enough to find a generating tuple close to the conjugated matrices $uau^*, ubu^*, uc^{(3)}u^*, \dots, uc^{(d)}u^*$. Note that uau^* is still diagonal, and that the entries $(ubu^*)_{i,n} = u_i b_{i,n} u_n^* = |b_{i,n}|$ are positive and invertible for $i = 1, \dots, n - 1$. Again, by considering the conjugated elements, we may from now on assume that a is diagonal, and $b_{1,n}, \dots, b_{n-1,n}$ of $b = c^{(2)}$ are positive, invertible.

Step 3: We show that we may also assume that the diagonal entries of a are invertible and have finite, disjoint spectra.

Let $a = \text{diag}(a_1, \dots, a_n)$ for elements $a_i \in A$. Since $\text{rr}(A) = 0$, we may perturb each a_i to be an invertible, self-adjoint element with finite spectrum. By perturbing the elements a_i further, we may assume that their spectra are also disjoint.

From now on, we will assume that $a = c^{(1)}$ and $b = c^{(2)}$ have the additional properties as explained in step 1-3.

Step 4: Let $\delta_0 > 0$ be smaller than the distance between any two points in $\sigma(a) \cup \{0\}$. For $k = 1, \dots, n$, define a continuous function $f_k: \mathbb{R} \rightarrow [0, 1]$ such that:

- (1) f_k has value 1 on a $\delta_0/4$ -neighborhood of $\sigma(a_k)$,
- (2) f_k has value 0 on a $\delta_0/4$ -neighborhood of $\bigcup_{i \neq k} \sigma(a_i) \cup \{0\}$

Let $\delta > 0$ be such that the spectrum $\sigma(a'_k)$ is contained in a $\delta_0/4$ -neighborhood of $\sigma(a_k)$ whenever $a'_k \in A_{\text{sa}}$ satisfies $a'_k =_\delta a_k$. In that case, we have

$$f_k(a_i) = \begin{cases} 1_A & , \text{ if } k = i \\ 0 & , \text{ if } k \neq i \end{cases},$$

and so we may recover the diagonal matrix units from such $a' = \text{diag}(a'_1, \dots, a'_n)$ as $1_A \otimes e_{k,k} = f_k(a')$. We may assume $\delta < \varepsilon$.

Step 5: We consider the following elements of A :

- (1) The n self-adjoint elements a_1, \dots, a_n ,
- (2) the $(n - 1)^2$ self-adjoint elements corresponding to the entries of the upper-left $(n - 1, n - 1)$ corner of b ,
- (3) the $(n - 1)$ positive elements $b_{1,n}, \dots, b_{n-1,n}$,

- (4) the self-adjoint element $b_{n,n}$,
- (5) and the $(d-2)n^2$ self-adjoint elements corresponding to the entries of the $d-2$ self-adjoint matrices $c^{(3)}, \dots, c^{(k)} \in A \otimes M_n$.

Together, this gives $n + (n-1)^2 + 1 + (d-2)n^2 = (d-1)n^2 - n + 2$ self-adjoint and $n-1$ positive elements of A .

Since $\text{gr}(A) \leq (d-1)n^2$, we may find $(d-1)n^2 + 1$ self-adjoint elements of A that together generate A , and which we collect as follows:

- (1) A diagonal, self-adjoint matrix \bar{a} ,
- (2) A self-adjoint matrix $\bar{b} \in A \otimes M_n$, whose entries $\bar{b}_{1,n}, \dots, \bar{b}_{n-1,n}$ are self-adjoint,
- (3) and $(d-2)$ self-adjoint matrices $\bar{c}^{(3)}, \dots, \bar{c}^{(k)} \in A \otimes M_n$.

such that $\bar{a} =_\delta a$, $\bar{b} =_\varepsilon b$ and $\bar{c}^{(k)} =_\varepsilon c^{(k)}$ for $k = 3, \dots, d$. For $i = 1, \dots, n-1$, we may ensure that $\bar{b}_{i,n}$ is positive and invertible by choosing it close enough to $b_{i,n}$.

Let $D := C^*(\bar{a}, \bar{b}, \bar{c}^{(3)}, \dots, \bar{c}^{(d)}) \subset A \otimes M_n$. We claim that $D = A \otimes M_n$.

As explained in Step 4, we have chosen δ such that we can recover the diagonal matrix units from \bar{a} as $1_A \otimes e_{k,k} = f_k(\bar{a}) \in D$ for $k = 1, \dots, n$. Following ideas from Olsen, Zame, [OZ76], we consider the elements $g_i := (1_A \otimes e_{i,i})b'(1_A \otimes e_{n,n})$ for $i = 1, \dots, n-1$. Note that $g_i = \bar{b}_{i,n} \otimes e_{i,n}$ is an element of D . Then:

$$g_i^* g_i = (\bar{b}_{i,n})^2 \otimes e_{n,n} \in D.$$

Since $(\bar{b}_{i,n})^2$ is positive and invertible, we have:

$$(\bar{b}_{i,n})^{-1} \otimes e_{n,n} \in C^*(g_i^* g_i) \subset D.$$

Then:

$$1_A \otimes e_{i,n} = g_i \cdot ((\bar{b}_{i,n})^{-1} \otimes e_{n,n}) \in D.$$

It follows that D contains all matrix units $1_A \otimes e_{i,j}$. Since the entries of the matrices $\bar{a}, \bar{b}, \bar{c}^{(3)}, \dots, \bar{c}^{(k)}$ generate A , we get $D = A \otimes M_n$, as desired. \square

Remark 3.7. Let us observe that Theorem 3.6 can be complemented by a partial converse inequality: Whenever A is a unital C^* -algebra, and $\text{Gen}_k(M_n(A))_{\text{sa}}$ is dense in $M_n(A)_{\text{sa}}^k$, then necessarily $\text{Gen}_{kn^2}(A)_{\text{sa}}$ is dense in $A_{\text{sa}}^{kn^2}$. It follows:

$$\left\lceil \frac{\text{gr}(A) + 1}{n^2} \right\rceil - 1 \leq \text{gr}(A \otimes M_n).$$

Thus, if A is a unital C^* -algebra with real rank zero and stable rank one, $\text{gr}(A) \geq 1$, and $n \in \mathbb{N}$, then:

$$\left\lceil \frac{\text{gr}(A) + 1}{n^2} \right\rceil - 1 \leq \text{gr}(A \otimes M_n) \leq \left\lceil \frac{\text{gr}(A)}{n^2} \right\rceil.$$

This can probably be improved.

Proposition 3.8. *Let A be a separable, real rank zero C^* -algebra that tensorially absorbs a UHF-algebra. Then $\text{gr}(A) \leq 1$.*

Proof. We will first reduce to the case that A is unital. Let $p_1, p_2, \dots \in A$ be an approximate unit of projections. Consider the corners $A_n := p_n A p_n$. Then each A_n is a unital, separable, real rank zero C^* -algebra that tensorially absorbs a UHF-algebra. If we can show $\text{gr}(A_n) \leq 1$, then $\text{gr}(A) \leq 1$ by Proposition 2.13.

So we may assume that A is unital. To simplify the proof, we will assume that A absorbs the 2^∞ UHF-algebra, denoted by M_{2^∞} . For other UHF-algebras, the proof is

analogous but notationally more involved. Since M_{2^∞} is strongly self-absorbing, see [TW07], there exists a $*$ -isomorphism $\Phi: A \rightarrow A \otimes M_{2^\infty}$ that is approximately unitarily equivalent to the inclusion $\iota: A \rightarrow A \otimes M_{2^\infty}$ given by $\iota(x) = x \otimes 1$. This means there exists a sequence of unitaries $u_n \in A \otimes M_{2^\infty}$ such that $\lim_n u_n \Phi(a) u_n^* = \iota(a)$ for all $a \in A$.

Since $A \cong A \otimes M_{2^\infty}$, it is enough to show $\text{gr}(A \otimes M_{2^\infty}) \leq 1$. We need to approximate any pair $a, b \in A \otimes M_{2^\infty}$ of self-adjoint elements by a pair that generates $A \otimes M_{2^\infty}$, and we first reduce the problem to the case that a, b lie in the image of ι .

As explained in the proof of Theorem 3.6, for any unitary $u \in A \otimes M_{2^\infty}$, it is enough to find generators close to the conjugated elements uau^*, ubu^* . Using the unitaries u_n implementing the approximate unitary equivalence between Φ and ι , we see that $u_n a u_n^* \rightarrow \iota(\Phi(a))$ and $u_n b u_n^* \rightarrow \iota(\Phi(b))$. If we can find generators a', b' close to $\iota(\Phi(a)), \iota(\Phi(b))$, say $a' =_\nu \iota(\Phi(a))$ and $b' =_\nu \iota(\Phi(b))$ for some $\nu > 0$, then for n large enough we have $u_n^* a' u_n =_\nu a$ and $u_n^* b' u_n =_\nu b$, and moreover the pair $u_n^* a' u_n$ and $u_n^* b' u_n$ generates $A \otimes M_{2^\infty}$.

So, let $a, b \in A$ be self-adjoint elements, and $\varepsilon > 0$ be given. We need to find a generating pair of self-adjoint elements $c, d \in A \otimes M_{2^\infty}$ such that $c =_\varepsilon a \otimes 1$ and $d =_\varepsilon b \otimes 1$. Since $\text{rr}(A) = 0$, we may assume that a is invertible and has finite spectrum, i.e., $a = \sum_i \lambda_i p_i$ for some $\lambda_i \in \mathbb{R} \setminus \{0\}$ and pairwise orthogonal projections $p_i \in A$ that sum to 1_A .

Let $x_1, x_2 \dots \in A$ be a sequence of positive elements that generates A and such that $\|x_k\| \leq \varepsilon/2^k$. Let μ the smaller than the distance between any two values in $\sigma(a) \cup \{0\}$. We may assume $\mu < \varepsilon$. We picture M_{2^∞} as $M_{2^\infty} = \bigotimes_{k=1}^\infty M_2$, and we let $e_{ij}^{(k)}$, $i, j = 1, 2$, be the matrix units of the k -th copy of M_2 . For $p \geq 1$, we let $1^{(\geq p)}$ denote the unit of the factor $\bigotimes_{k=p}^\infty M_2$. Define $c, d \in A \otimes M_{2^\infty}$ as:

$$c := \sum_{k \geq 1} \left[\left(\sum_i \left(\lambda_i + \frac{\mu}{2^k} \right) p_i \right) \otimes e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes e_{11}^{(k)} \otimes 1^{(\geq k+1)} \right],$$

$$d := b \otimes 1 + \sum_{k \geq 1} x_k \otimes e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes \left(e_{12}^{(k)} + e_{21}^{(k)} \right) \otimes 1^{(\geq k+1)}$$

One checks $c =_\varepsilon a \otimes 1$ and $d =_\varepsilon b \otimes 1$. Set $D := C^*(c, d) \subset A \otimes M_{2^\infty}$, and let us check $D = A \otimes M_{2^\infty}$.

For $k \geq 1$, let $f_k, g_k: \mathbb{R} \rightarrow [0, 1]$ be continuous functions such that:

- (1) f_k takes value 1 on $\{\lambda_i + \frac{\mu}{2^k} \mid i \geq 1\}$, and value 0 on $\{\lambda_i + \frac{\mu}{2^l} \mid i \geq 1, l \neq k\} \cup \{0\}$.
- (2) g_k takes value 1 on $\{\lambda_i + \frac{\mu}{2^l} \mid i \geq 1, l \neq k\}$, and value 0 on $\{\lambda_i + \frac{\mu}{2^k} \mid i \geq 1\} \cup \{0\}$.

Then

$$e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes e_{11}^{(k)} \otimes 1^{(\geq k+1)} = f_k(c) \in D,$$

$$e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} = g_k(c) \in D.$$

We follow ideas of Olsen, Zame, [OZ76]. For $k \geq 1$, consider the element $y_k := f_k(c) d g_k(c) \in D$. By construction of d , we have:

$$y_k = x_k \otimes e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes e_{12}^{(k)} \otimes 1^{(\geq k+1)}.$$

Then

$$y_k^* y_k = x_k^2 \otimes e_{22}^{(1)} \otimes \dots \otimes e_{22}^{(k-1)} \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} \in D.$$

Since x_k^2 is positive and invertible, we have $(x_k)^{-1} \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} \in D$, and then:

$$1_A \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{12}^{(k)} \otimes 1^{(\geq k+1)} \in D.$$

It follows that D contains all matrix units of $1 \otimes M_{2^\infty}$, and therefore $1 \otimes M_{2^\infty} \subset D$. Then $x_k \otimes 1 \in D$ for all k . Since the x_k generate A , we get $A \otimes 1 \subset D$, and then $D = A \otimes M_{2^\infty}$, as desired. \square

Remark 3.9. We do not know of any example of a separable, real rank zero C^* -algebra A for which $\text{gr}(A) \geq 2$, and it is possible that no such algebra exists. Let us consider the following weaker question: Do all separable, unital, simple, real rank zero, stable rank one C^* -algebras have generator rank at most one, or are they at least singly generated?

This is certainly a hard problem, since a positive answer to it would give a positive solution to the generator problem for von Neumann algebras, which asks whether every von Neumann algebra acting on a separable Hilbert space is singly generated, see [Kad67, Problem 14] and [Ge03]. The generator problem for von Neumann algebras has been reduced to the case of a type II_1 -factor, see [Wil74]. Every II_1 -factor M acting on a separable Hilbert space contains a separable unital, simple, real rank zero, stable rank one C^* -algebras $A \subset M$ such that $A'' = M$. If A is singly generated (as a C^* -algebra), then so is M (as a von Neumann algebra).

4. THE GENERATOR RANK OF HOMOGENEOUS C^* -ALGEBRAS

In this section, we first compute the generator rank of commutative C^* -algebras, see Proposition 4.7. The main result is Theorem 4.23, which shows how to compute the generator rank of homogeneous C^* -algebras. To obtain these results, we have to compute the codimension of $\text{Gen}_k(M_n)_{\text{sa}} \subset (M_n)_{\text{sa}}^k$, see Lemma 4.20.

For spaces X, Y we denote by $E(X, Y)$ the space of continuous embeddings.

4.1. Let us discuss the generator rank of commutative C^* -algebras.

Let X be a compact, metric space, and $k \in \mathbb{N}$. We may identify $C(X)_{\text{sa}}^{k+1} \cong C(X, \mathbb{R}^{k+1})$. By the Stone-Weierstrass theorem, an element $\mathbf{a} \in C(X)_{\text{sa}}^{k+1}$ generates $C(X)$ if and only if $\mathbf{a}(x) = (a_1(x), \dots, a_{k+1}(x)) \neq 0$ for all $x \in X$, and \mathbf{a} separates the points of X .

It follows that $\text{gr}(C(X)) \leq k$ if and only if $E(X, \mathbb{R}^{k+1} \setminus \{0\})$ is dense in $C(X, \mathbb{R}^{k+1})$.

We will break this problem into two parts:

- (1) When is $C(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1})$ dense?
- (2) When is $E(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1} \setminus \{0\})$ dense?

These questions have been studied and answered in a more general setting. We recall the results in a way that will also be used to compute the generator rank of homogeneous C^* -algebras.

4.2. Let X be a compact, metric space, let $q \geq 1$. We give $C(X, \mathbb{R}^q)$ the topology induced by the supremum norm $\|f\| := \sup_{x \in X} |f(x)|$.

If $Y \subset \mathbb{R}^q$ is closed, then it follows from compactness of X that $C(X, \mathbb{R}^q \setminus Y)$ is an open subset of $C(X, \mathbb{R}^q)$. We want to see when it is also dense. The goal is Proposition 4.4 below, which is a classical result that can be obtained in several ways. A particular version appeared in [BE91, Theorem 1.3].

Recall that $Y \subset \mathbb{R}^q$ is said to be “codimension three” if $\dim(Y) \leq q-3$. For the notion of “tameness” of embeddings we refer the reader to the survey by Edwards, [Edw75]. We

note that every codimension three submanifold $M \subset \mathbb{R}^q$, $q \geq 5$, is tamely embedded. It follows that $Y \subset \mathbb{R}^q$ is a codimension three, tame embedding, if Y is a countable union of codimension three submanifolds of \mathbb{R}^q , $q \geq 5$. To derive Proposition 4.4, we will use the following result:

Proposition 4.3 (Dranishnikov, [Dra91]). *Let X be a compact, metric space, and $Y \subset \mathbb{R}^q$ be a codimension three, tame compact subset. Then the following are equivalent:*

- (1) $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is dense,
- (2) $\dim(X \times Y) < q$.

Proposition 4.4. *Let X be a compact, metric space, let $q \geq 5$, and let $Y \subset \mathbb{R}^q$ be a closed subset that is the countable union of codimension three submanifolds of \mathbb{R}^q . Then the following are equivalent:*

- (1) $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is dense (and open),
- (2) $\dim(X) < q - \dim(Y)$.

Proof. We noted in 4.2 that $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is always open.

Set $d := \dim(Y)$. Let $Y_1, Y_2, \dots \subset \mathbb{R}^q$ be codimension three submanifolds such that $Y = \bigcup_k Y_k$. Then $\dim(Y) = \sup_k \dim(Y_k)$, and so there exist an index l with $\dim(Y_l) = d$. Choose a compact subset $Z \subset Y_l$ such that $Z \cong [0, 1]^d$. It follows from [Edw75] that Z is tamely embedded in \mathbb{R}^q .

Let us show “(1) \Rightarrow (2)”. Since $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q \setminus Z)$, we get $\dim(X \times Z) < q$ from Proposition 4.3. Then (2) follows, since $\dim(X \times [0, 1]^d) = \dim(X) + d$.

To show “(2) \Rightarrow (1)”, assume $\dim(X) < q - \dim(Y)$. Let $f \in C(X, \mathbb{R}^q)$, and $\varepsilon > 0$ be given. We let $B \subset \mathbb{R}^q$ be the closed ball of radius $(1 + \varepsilon)\|f\|$. Then $Y \cap B$ is codimension three, tame compact subset. Moreover, $\dim(X \times (Y \cap B)) \leq \dim(X) + \dim(Y) < q$. By Proposition 4.3, we may find $g \in C(X, \mathbb{R}^q \setminus (Y \cap B))$ with $g =_\varepsilon f$. By construction, $g \in C(X, \mathbb{R}^q \setminus Y)$. Since f and ε were arbitrary, this shows that $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is dense, as desired. \square

4.5. Let X be a compact, metric space with $\dim(X) = d$. It is a classical result that X can be embedded into \mathbb{R}^{2d+1} , and even more, the embeddings $E(X, \mathbb{R}^{2d+1})$ are dense in $C(X, \mathbb{R}^{2d+1})$, see e.g. [Eng95, Theorem 1.11.4, p.95].

The converse is not quite true, and it is connected to the question whether $\dim(X \times X) = 2 \dim(X)$. It is known that the dimension of $X \times X$ can only take the values $2 \dim(X) - 1$ or $2 \dim(X)$. Thus, we may divide the class of all compact, metric, finite-dimensional spaces into two classes, see the Definition before 3.17 in [Dra01]:

- (1) If $\dim(X \times X) = 2 \dim(X)$, we say X is “of basic type”.
- (2) If $\dim(X \times X) < 2 \dim(X)$, we say X is “of exceptional type”.

If $\dim(X) \leq 1$, then X is of basic type, and for every $d \geq 2$ there exists a space X of exceptional type with $\dim(X) = d$.

It is shown by Spiež, [Spi90, Theorem 2], that X is of exceptional type if and only if $E(X, \mathbb{R}^{2d})$ is dense in $C(X, \mathbb{R}^{2d})$. As shown in [DRS91, Theorem 1.1], one may deduce that the following are equivalent:

- (1) $\dim(X \times X) < q$.
- (2) $E(X, \mathbb{R}^q) \subset C(X, \mathbb{R}^q)$ is dense.

This result was generalized to topological manifolds by Luukkainen. By a q -manifold we mean a separable, metric space M such that every point $x \in M$ has a neighborhood homeomorphic to \mathbb{R}^q . It follows from [Luu81, Theorem 5.1] that $E(C, M) \subset C(X, M)$ is

dense if $q \geq 2d + 1$. This is complemented by [Luu91, Theorem 2.5] which shows that X is of exceptional type if and only if $E(X, M) \subset C(X, M)$ is dense for some (and hence all) $2d$ -manifolds.

This shows the following result:

Proposition 4.6 (Lukkainen, [Luu81], [Luu91]). *Let X be a compact, metric space, and M a manifold. Then the following are equivalent:*

- (1) $\dim(X \times X) < \dim(M)$,
- (2) $E(X, M) \subset C(X, M)$ is dense.

Proposition 4.7. *Let X be a compact, metric space. Then $\text{gr}(C(X)) = \dim(X \times X)$. More generally, if A is a separable, commutative C^* -algebra, then*

$$\text{gr}(A) = \dim(\text{Prim}(A) \times \text{Prim}(A)).$$

Proof. Let X be a compact, metric space, and let $k \in \mathbb{N}$. As explained in 4.1, we have $\text{gr}(C(X)) \leq k$ if and only if the following conditions hold:

- (1) $C(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1})$ is dense.
- (2) $E(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1} \setminus \{0\})$ is dense.

It follows from Proposition 4.4 that (1) is equivalent to $\dim(X) \leq k$, and it follows from Proposition 4.6 that (2) is equivalent to $\dim(X \times X) \leq k$.

Since $\dim(X) \leq \dim(X \times X)$, condition (2) implies condition (1), and we deduce:

$$\text{gr}(C(X)) \leq k \iff \dim(X \times X) \leq k,$$

from which the result follows.

Now let A be a non-unital, separable, commutative C^* -algebra, and set $X := \text{Prim}(A)$. Then \tilde{A} has primitive ideal space αX , the one-point compactification of X . Then $\text{gr}(A) = \text{gr}(\tilde{A}) = \dim((\alpha X) \times (\alpha X))$, and the result follows since $\dim((\alpha X) \times (\alpha X)) = \dim(X \times X)$. \square

Lemma 4.8. *Let X, Y be compact, metric spaces. Set $Z = X \sqcup Y$. Then:*

$$\dim(Z \times Z) \leq \max\{\dim(X \times X), \dim(Y \times Y)\}.$$

Moreover, if both X and Y are of exceptional type (see 4.5), then so is $Z = X \sqcup Y$.

Proof. For any compact, metric space M , we use the notation $M^k := M \times \dots_k \times M$ for the k -fold Cartesian power. It is shown in [Dra01, Theorem 3.16] that M is of exceptional type if and only if $\dim(M^k) = k \dim(X) - k + 1$ for $k \geq 1$, and that M is of basic type if and only if $\dim(M^k) = k \dim(X)$ for $k \geq 1$. If M, N are two compact spaces, then $\dim(M \times N) \leq \dim(M) + \dim(N)$, by the product theorem of covering dimension.

Now let X, Y be two compact, metric space, and set $Z = X \sqcup Y$. Note that we have $\dim(Z) = \max\{\dim(X), \dim(Y)\}$. We distinguish two cases:

Case 1: Assume $\dim(X) \neq \dim(Y)$. Without loss of generality we may assume $\dim(X) < \dim(Y)$. Then $\dim(Z) = \dim(Y)$. Moreover, $\dim(X \times Z) \leq \dim(X) + \dim(Y) \leq \dim(Y^2)$, and $\dim(X^2) \leq 2 \dim(X) \leq \dim(Y^2)$. Thus, we may estimate:

$$\dim(Z^2) = \max\{\dim(X^2), \dim(X \times Y), \dim(Y^2)\} \leq \dim(Y^2),$$

and we also obtain the desired inequality $\dim(Z^2) \leq \max\{\dim(X^2), \dim(Y^2)\}$.

If Y is of exceptional type, then

$$\dim(Z^2) \leq \dim(Y^2) = 2 \dim(Y) - 1 = 2 \dim(Z) - 1,$$

showing that Z is of exceptional type.

Case 2: Assume $\dim(X) = \dim(Y)$, and set $d := \dim(X)$. We may estimate:

$$\begin{aligned} \dim(Z^2) &= \max\{\dim(X^2), \dim(X \times Y), \dim(Y^2)\} \\ &\leq \max\{2\dim(X), \dim(X) + \dim(Y), 2\dim(Y)\} \\ &\leq 2d. \end{aligned}$$

If at least one of X or Y is of basic type, then $\max\{\dim(X^2), \dim(Y^2)\} = 2d$, showing the desired inequality.

If both X and Y are of exceptionnal type, then $\dim(X^2) = \dim(Y^2) = 2d - 1$, and we have the following estimate:

$$\dim(X^2 \times Y) \leq \dim(X^2) + \dim(Y) = 3d - 1,$$

and similarly $\dim(X \times Y^2) \leq 3d - 1$.

Then:

$$\begin{aligned} \dim(Z^3) &= \max\{\dim(X^3), \dim(X^2 \times Y), \dim(X \times Y^2), \dim(Y^3)\} \\ &\leq \max\{3d - 2, 3d - 1, 3d - 1, 3d - 2\} \\ &\leq 3d - 1. \end{aligned}$$

If Z were of basic type, then $\dim(Z^3) = 3\dim(Z)$. Thus, Z is of exceptional type, and so $\dim(Z^2) = 2\dim(Z) - 1 = 2d - 1 = \max\{\dim(X^2), \dim(Y^2)\}$. \square

Proposition 4.9. *Let A and B be separable, commutative C^* -algebras. Then $\text{gr}(A \oplus B) = \max\{\text{gr}(A), \text{gr}(B)\}$.*

Proof. We have $\text{gr}(A), \text{gr}(B) \leq \text{gr}(A \oplus B)$ by Proposition 2.12. For the converse inequality, note that $A \oplus B$ is an ideal in $\tilde{A} \oplus \tilde{B}$. Then, using Theorem 2.14 at the first step, using Proposition 4.7 and Lemma 4.8 at the second step:

$$\text{gr}(A \oplus B) \leq \text{gr}(\tilde{A} \oplus \tilde{B}) \leq \max\{\text{gr}(\tilde{A}), \text{gr}(\tilde{B})\} = \max\{\text{gr}(A), \text{gr}(B)\},$$

as desired. \square

We now turn to the computation of the generator rank of homogeneous C^* -algebras. We first recall the well-known structure theory of such algebras.

Definition 4.10 (Fell, [Fel61, 3.2]). Let A be a C^* -algebra and $n \geq 1$. Then A is called **n -homogeneous** if all its irreducible representations are n -dimensional. We further say that A is **homogeneous** if it is n -homogeneous for some n .

4.11. Let us recall a general construction: Assume $\mathfrak{B} = (E \xrightarrow{p} X)$ is a locally trivial fibre bundle (over a locally compact, Hausdorff space X) whose fiber has the structure of a C^* -algebra. Let

$$(4.1) \quad \Gamma_0(\mathfrak{B}) = \{f: X \rightarrow E \mid p \circ f = \text{id}_X, (x \rightarrow \|f(x)\|) \in C_0(X)\}$$

be the sections of \mathfrak{B} that vanish at infinity. Then $\Gamma_0(\mathfrak{B})$ has a natural structure of a C^* -algebra, with the algebraic operations defined fibrewise, and norm $\|f\| := \sup_{x \in X} \|f(x)\|$.

If the bundle has fibre M_n (a so-called M_n -bundle), then $A := \Gamma_0(\mathfrak{B})$ is n -homogeneous and $\text{Prim}(A) \cong X$. Thus, every M_n -bundle defines an n -homogeneous C^* -algebra. The converse does also hold:

Proposition 4.12 (Fell, [Fel61, Theorem 3.2]). *Let A be a C^* -algebra, $n \in \mathbb{N}$. Then the following are equivalent:*

- (1) A is n -homogeneous,
- (2) $A \cong \Gamma_0(\mathfrak{B})$ for a locally trivial M_n -bundle \mathfrak{B} .

4.13. Let A be a n -homogeneous C^* -algebra, and set $X := \text{Prim}(A)$. Then A is naturally a $C_0(X)$ -algebra, with each fiber isomorphic to M_n . For the definition and results of $C_0(X)$ -algebras, we refer the reader to § 1 of [Kas88] or § 2 of [Dad09]. We use the same notation as in [TW12, 2.4].

4.14. Let A be a C^* -algebra, and $k \in \mathbb{N}$. Let us denote the automorphism group of A by $\text{Aut}(A)$. We define a natural action Υ of $\text{Aut}(A)$ on A_{sa}^k . Given $\alpha \in \text{Aut}(A)$ and $\mathbf{a} = (a_1, \dots, a_k) \in A_{\text{sa}}^k$ we set:

$$\Upsilon(\alpha)(\mathbf{a}) = \alpha \cdot \mathbf{a} := (\alpha(a_1), \dots, \alpha(a_k)).$$

For $\alpha \in \text{Aut}(A)$ and $\mathbf{a} \in A_{\text{sa}}^k$, note that $\alpha \cdot \mathbf{a} = \mathbf{a}$ if and only if $\alpha(x) = x$ for all $x \in C^*(\mathbf{a})$. Thus, the restriction of Υ to $\text{Gen}_k(A)_{\text{sa}}$ is free.

For $A = M_n$, every automorphism is inner. The kernel of the map $U_n \rightarrow \text{Aut}(M_n)$ is the group of central unitary matrices $\mathbb{T} \cdot 1 \subset U_n$. Let $PU_n := U_n/(\mathbb{T} \cdot 1)$ be the projective unitary group. It follows that $\text{Aut}(M_n) \cong PU_n$, which is a compact Lie group of dimension $n^2 - 1$.

Lemma 4.15. *Let A be a simple C^* -algebra, let $k \in \mathbb{N}$, and let $\mathbf{a}, \mathbf{b} \in \text{Gen}_k(A)_{\text{sa}}$. Then $\mathbf{a} \oplus \mathbf{b} \in \text{Gen}_k(A \oplus A)_{\text{sa}}$ if and only if \mathbf{a} and \mathbf{b} lie in different orbits of the action of $\text{Aut}(A)$ on $\text{Gen}_k(A)_{\text{sa}}$.*

Proof. If $\mathbf{a} = \alpha \cdot \mathbf{b}$ for some $\alpha \in \text{Aut}(A)$, then $C^*(\mathbf{a} \oplus \mathbf{b}) = \{(x, \alpha(x)) \mid x \in A\} \cong A$, and so $\mathbf{a} \oplus \mathbf{b}$ does not generate $A \oplus A$.

Conversely, let $D := C^*(\mathbf{a} \oplus \mathbf{b}) \subset A \oplus A$, and assume that $D \neq A \oplus A$. Let $\pi_i: A \oplus A \rightarrow A$ be the surjective morphisms on the two summands, $i = 1, 2$. Then $\pi_1(D) = A$. Note that $\ker(\pi_1) = 0 \oplus A$. If $\ker(\pi_1) \cap D \neq 0$, then $\ker(\pi_1) \cap D = 0 \oplus A$, and so $D = A \oplus A$, a contradiction. Therefore, $\pi_1: D \rightarrow A$ is an isomorphism. Similarly $\pi_2: D \rightarrow A$ is an isomorphism. Then $\alpha := \pi_2 \circ \pi_1^{-1}$ is the desired automorphism of A that satisfies $\alpha \cdot \mathbf{a} = \mathbf{b}$. \square

Proposition 4.16. *Let X be a compact, metric space, let A be a simple, separable C^* -algebra, and let B a continuous $C(X)$ -algebra with fibers isomorphic to A . Let $k \in \mathbb{N}$, and $\mathbf{b} \in B_{\text{sa}}^k$. Then $\mathbf{b} \in \text{Gen}_k(B)_{\text{sa}}$ if and only if the following two conditions are satisfied:*

- (1) \mathbf{b} pointwise generates A , i.e., for each $x \in X$ we have $\mathbf{b}(x) = (b_1(x), \dots, b_k(x)) \in \text{Gen}_k(A)_{\text{sa}}$.
- (2) \mathbf{b} separates the points of X in the sense that for distinct $x, y \in X$ the tuples $\mathbf{b}(x)$ and $\mathbf{b}(y)$ lie in different orbits of the action of $\text{Aut}(A)$ on $\text{Gen}_k(A)_{\text{sa}}$.

Proof. Let us first assume that $\mathbf{b} \in \text{Gen}_k(B)_{\text{sa}}$. For $x \in X$, let $\pi_x: B \rightarrow A$ be the surjective morphism to the fiber at x . This maps $\text{Gen}_k(B)_{\text{sa}}$ into $\text{Gen}_k(A)_{\text{sa}}$. This shows (1). For distinct points $x, y \in X$, consider the surjective morphism $\varphi := \pi_x \oplus \pi_y: B \rightarrow A \oplus A$. Since φ maps $\text{Gen}_k(B)_{\text{sa}}$ into $\text{Gen}_k(A \oplus A)_{\text{sa}}$, we get $\mathbf{b}(x) \oplus \mathbf{b}(y) = \varphi(\mathbf{b}) \in \text{Gen}_k(A \oplus A)_{\text{sa}}$, and so (2) follows from Lemma 4.15.

The converse follows from [TW12, Lemma 3.2], which is proved using the factorial Stone-Weierstrass conjecture. \square

Notation 4.17. For $n, k \in \mathbb{N}$, set:

$$E_n^k := (M_n)_{\text{sa}}^k, \quad G_n^k := \text{Gen}_k(M_n)_{\text{sa}} \subset E_n^k.$$

Lemma 4.18. *Let $n, k \in \mathbb{N}$. Then $G_n^k \subset E_n^k$ is open.*

Proof. In general, if A is any C^* -algebra, and $\text{Gen}_l(A)_{\text{sa}}$ contains a non-empty open subset, for some l , then $\text{Gen}_k(A)_{\text{sa}}$ is open (but possibly empty) for all k . To see this, let $U \subset \text{Gen}_l(A)_{\text{sa}}$ be a non-empty, open subset. Let \mathbf{a} be any element of $\text{Gen}_k(A)_{\text{sa}}$. Then there exist polynomials p_1, \dots, p_l such that $(p_1(\mathbf{a}), \dots, p_l(\mathbf{a})) \in U$. For $i = 1, \dots, l$, let $q_i := (p_i + p_i^*)/2$, considered as a polynomial that maps A_{sa}^k to A_{sa} . Consider the continuous map $q: A_{\text{sa}}^k \rightarrow A_{\text{sa}}^l$ given by $q(\mathbf{x}) := (q_1(\mathbf{x}), \dots, q_l(\mathbf{x}))$. Then $q(\mathbf{a}) \in U$, and so $q^{-1}(U)$ is an open subset of $\text{Gen}_k(A)_{\text{sa}}$ containing \mathbf{a} .

Let $e_{i,j} \in M_n$ denote the matrix units. Consider the self-adjoint element $a = \sum_{s=1}^n \frac{s}{n} e_{s,s}$, and $b = \sum_{s=1}^{n-1} (e_{s,s+1} + e_{s+1,s})$. Then every pair $(a', b') \in (M_n)_{\text{sa}}^2$ close enough to (a, b) generates M_n . Indeed, if a' is close enough to a , then it generates a maximal abelian subalgebra of M_n . If b' is close enough to b , then it does not commute with the elements of $C^*(a')$, and it follows that the commutant of $C^*(a', b')$ is $\mathbb{C} \cdot 1$. This implies $C^*(a', b') = M_n$, as desired. \square

Lemma 4.19. *Let $n, k \in \mathbb{N}$. Let Υ denote the action of PU_n on E_n^k , as defined in 4.14. Let $\mathbf{a} \in E_n^k$. Then the following are equivalent:*

- (1) *The stabilizer subgroup of \mathbf{a} is trivial.*
- (2) *$\mathbf{a} \in G_n^k$.*

Proof. Let $\alpha \in PU_n$, and let it be represented by a unitary $u \in M_n$. It was noted in 4.14 that $\alpha \cdot \mathbf{a} = \mathbf{a}$ if and only if $\alpha(x) = x$ for all $x \in C^*(\mathbf{a})$. This shows the implication “(2) \Rightarrow (1)”.

For the converse, assume $\mathbf{a} \in E_n^k \setminus G_n^k$. Let $B := C^*(\mathbf{a})$, the sub- C^* -algebras generated by \mathbf{a} . Since B is a proper sub- C^* -algebra, the commutant B' is non-trivial. It follows that there exists a unitary $u \in B'$ which induces a non-trivial automorphism on M_n , while it stabilizes \mathbf{a} , and so the stabilizer subgroup of \mathbf{a} is non-trivial. \square

Lemma 4.20. *Let $n, k \geq 2$. Then $E_n^k \setminus G_n^k$ is a closed subset which is a finite union of codimension three submanifolds and $\dim(E_n^k \setminus G_n^k) = kn^2 - (k-1)(2n-2)$.*

If, moreover, X is a compact, metric space, then the following are equivalent:

- (1) *$C(X, G_n^k) \subset C(X, E_n^k)$ is dense (and open),*
- (2) *$\dim(X) < (k-1)(2n-2)$.*

Proof. Set $Y := E_n^k \setminus G_n^k$, which is closed by Lemma 4.18. The action Υ of PU_n on E_n^k , as defined in 4.14, is a smooth action of a compact Lie group on a manifold, and we consider its orbit type decomposition.

In general, if a compact Lie group G acts smoothly on a manifold M , and $H \leq G$ is a closed subgroup, then $M_{(H)}$ denotes the set of points $x \in M$ such that the stabilizer subgroup of x is conjugate to H . Then M decomposes into orbit types $M = \bigcup M_{(H)}$. Let $\pi: M \rightarrow M/G$ denote the quotient map to the orbit space. Then, for each H , the image of $\pi(M_{(H)}) \subset M/G$ has a unique manifold structure, and the restriction of π to $M_{(H)}$ is a submersion. We refer the reader to [Mei03] for more details.

By Lemma 4.19, G_n^k is the submanifold corresponding to the trivial stabilizer subgroup. Thus, we have to consider $M_{(H)}$ for subgroups $H \leq PU_n$ with $H \neq \{1\}$.

Let $V(M_n)$ denote the space of sub- C^* -algebras of M_n . Then PU_n naturally acts on $V(M_n)$. Let $\Psi: E_n^k \rightarrow V(M_n)$ be the map that sends \mathbf{a} to $\Psi(\mathbf{a}) := C^*(\mathbf{a}) \subset M_n$. It is easily checked that Ψ is PU_n -equivariant.

There are only finitely many orbits of the action of PU_n on $V(M_n)$, which we may label by multi-indices $\omega = (d_1, m_2, \dots, d_s, m_s)$ for $d_i, m_i \in \mathbb{N}$ and $\sum_i m_i d_i \leq n$. The multi-index $\omega = (d_1, m_2, \dots, d_s, m_s)$ is assigned to a sub- C^* -algebra $B \subset M_n$ if $B \cong M_{d_1} \oplus \dots \oplus M_{d_s}$, and the copy $M_{d_i} \subset B$ has multiplicity m_i in the embedding $B \subset M_n$. Two indices can be assigned to the same orbit, and one could put more restrictions on the indices to get a unique assignment. For our considerations, however, this is unimportant, since we are only interested in the dimension of certain submanifolds, and it is no problem if we consider the same submanifold several times.

Let $\omega = (d_1, m_2, \dots, d_s, m_s)$ be a multi-index, and let $B \subset M_n$ be a sub- C^* -algebra whose orbit in $V(M_n)$ has index ω . For $\mathbf{a} \in E_n^k$, we have $\Psi(\mathbf{a}) = B$ if and only if $\mathbf{a} \in \text{Gen}_k(B)_{\text{sa}} \subset \text{Gen}_k(M_n)_{\text{sa}} = E_n^k$. By Corollary 3.3, we have $\text{gr}(B) \leq 1$, and since $k \geq 2$ we get that $\text{Gen}_k(B)_{\text{sa}} \subset B_{\text{sa}}^k$ is dense. Moreover, it follows from Lemma 4.18 that $\text{Gen}_k(B)_{\text{sa}} \subset B_{\text{sa}}^k$ is open. Thus, $\Psi^{-1}(B) = \text{Gen}_k(B)_{\text{sa}}$ is a dense, open subset of the manifold B_{sa}^k , and therefore:

$$\dim(\Psi^{-1}(B)) = \dim(B_{\text{sa}}^k) = k \sum_i d_i^2.$$

Let $K(B) \subset PU_n$ be the stabilizer subgroup of B . If B is a unital subalgebra of M_n , then every unitary of B stabilizes B , and so we obtain a natural map $U(B) \rightarrow K(B)$ with kernel $U(B) \cap \mathbb{T} \cdot 1$. If B is a non-unital subalgebra, let $r \in M_n$ denote the projection such that $1_B + r = 1$. For every unitary $u \in B$, the element $u + r$ is a unitary in M_n that stabilizes B . Thus, in the unital case, $K(B)$ has a subgroup isomorphic to $U(B)/\mathbb{T}$, and in the non-unital case, $K(B)$ has a subgroup isomorphic to $U(B)$. Therefore, we get the rough (but for our purposes sufficient) estimate:

$$\dim(K(B)) \geq \dim U(B) - 1 = \sum_i d_i^2 - 1.$$

Let $V(\omega) \subset V(M_n)$ be the orbit with index ω . Then:

$$\dim(V(\omega)) = \dim(PU_n) - \dim(K(B)) \leq n^2 - 1 - [\sum_i d_i^2 - 1] = n^2 - \sum_i d_i^2.$$

Therefore:

$$\begin{aligned} \dim(\Psi^{-1}(V(\omega))) &= \dim(\Psi^{-1}(B)) + \dim(V(\omega)) \\ &\leq k \sum_i d_i^2 + n^2 - \sum_i d_i^2 \\ &= n^2 + (k-1) \sum_i d_i^2. \end{aligned}$$

Note that $\omega = (n, 1)$ labels the one-point orbit of $M_n \subset M_n$, and so $\Psi^{-1}(V(n, 1)) = G_n^k$. Among $\omega \neq (n, 1)$, $n^2 + (k-1) \sum_i d_i^2$ has its maximum value for the partition $d_1 = n-1, d_2 = 1$ (and $m_1 = m_2 = 1$). Thus, for $\omega \neq (n, 1)$:

$$\dim(\Psi^{-1}(V(\omega))) \leq n^2 + (k-1)[(n-1)^2 + 1] = kn^2 - (k-1)(2n-2).$$

Since $E_n^k \setminus G_n^k = \bigcup_{\omega \neq (n, 1)} \Psi^{-1}(V(\omega))$, we get:

$$\dim(E_n^k \setminus G_n^k) \leq kn^2 - (k-1)(2n-2).$$

The partition $\omega' := (n-1, 1, 1, 1)$ labels the orbit of $B = M_{n-1} \oplus \mathbb{C} \subset M_n$, and one checks $K(B) \cong U_{n-1}$, so that $\dim(V(\omega')) = n^2 - 1 - (n-1)^2$ and then $\dim(\Psi^{-1}(V(\omega'))) = kn^2 - (k-1)(2n-2)$. Since $\Psi^{-1}(V(\omega')) \subset E_n^K \setminus G_n^k$, we get $\dim(E_n^K \setminus G_n^k) \geq kn^2 - (k-1)(2n-2)$, and so $\dim(E_n^K \setminus G_n^k) = kn^2 - (k-1)(2n-2)$, as desired.

Now the assertion of the equivalent conditions follows from our dimension computations and Proposition 4.4. \square

Lemma 4.21. *Let A be a unital, separable n -homogeneous C^* -algebra, $n \geq 2$, and let $k \geq 2$. Let $X := \text{Prim}(A)$, and for $x \in X$ let $\pi_x: A \rightarrow M_n$ be a morphism onto the fiber at x . Let S be the set of elements $\mathbf{a} \in A_{\text{sa}}^k$ such that for all $x \in \text{Prim}(A)$, the tuple $\pi_x(a_1), \dots, \pi_x(a_k)$ generates M_n . Then the following are equivalent:*

- (1) $S \subset A_{\text{sa}}^k$ is dense (and open),
- (2) $\dim(X) < (k-1)(2n-2)$.

Proof. There are finitely many closed subsets $X_1, \dots, X_r \subset X$ such that the M_n -bundle associated to A is trivial over each X_i . For each i , let $\Phi_i: A \rightarrow C(X_i, M_n)$ be a trivialization. This induces a surjective map $A_{\text{sa}}^k \rightarrow C(X_i, M_n)_{\text{sa}}^k \cong C(X_i, E_n^k)$, we also denote by Φ_i , and which is open by then open mapping theorem. Note that $\Phi_i(S) = C(X_i, G_n^k)$.

It follows that $S = \bigcap_i \Phi_i^{-1}(C(X_i, G_n^k))$ is dense if and only if each $C(X_i, G_n^k) \subset C(X_i, E_n^k)$ is dense. By Lemma 4.20, this is equivalent to $\dim(X_i) < (k-1)(2n-2)$ for each i , and since the X_i form a finite, closed cover of X , this is equivalent to $\dim(X) < (k-1)(2n-2)$. \square

4.22. Let A be a unital, separable n -homogeneous C^* -algebra, and let $k \in \mathbb{N}$. Let $X := \text{Prim}(A)$, and for $x \in X$ let $\pi_x: A \rightarrow M_n$ be a morphism onto the fiber at x . This induces a natural map $A_{\text{sa}}^k \rightarrow E_n^k$, which we also denote by π_x .

Let us define a map $\Psi: A_{\text{sa}}^k \rightarrow C(X, E_n^k/PU_n)$ by $\Psi(\mathbf{a})(x) := PU_n \cdot \pi(\mathbf{a})$. Note that the map from A to its fiber at some point $x \in X$ is not unique. It is, however, unique up to an automorphism of M_n , which shows that Ψ is well-defined.

Restricting Ψ to the subset S as defined in Lemma 4.21, gives a map $\Psi: S \rightarrow C(X, G_n^k/PU_n)$. Proposition 4.16 shows that $\text{Gen}_k(A)_{\text{sa}} = \Psi^{-1}(E(X, G_n^k/PU_n))$.

Theorem 4.23. *Let A be a unital, separable n -homogeneous C^* -algebra, $n \geq 2$. Set $X := \text{Prim}(A)$, the primitive ideal space of A . Then:*

$$\text{gr}(A) = \left\lceil \frac{\dim(X) + 1}{2n-2} \right\rceil.$$

Proof. Since M_n is a quotient of A , we get $\text{gr}(A) \geq \text{gr}(M_n) = 1$ by Proposition 2.12 and Lemma 3.2. We also have $\left\lceil \frac{\dim(X)+1}{2n-2} \right\rceil \geq 1$ for every value of $\dim(X)$. Thus, it is enough to show that for every $k \geq 2$ the following holds:

$$\text{gr}(A) \leq k-1 \iff \dim(X) < (k-1)(2n-2).$$

Let $S \subset A_{\text{sa}}^k$ be defined as in Lemma 4.21.

Assume $\text{gr}(A) \leq k-1$, i.e., $\text{Gen}_k(A)_{\text{sa}} \subset A_{\text{sa}}^k$ is dense. Since $\text{Gen}_k(A)_{\text{sa}} \subset S$, it follows from Lemma 4.21 that $\dim(X) < (k-1)(2n-2)$.

Conversely, assume $\dim(X) < (k-1)(2n-2)$. Again by Lemma 4.21, we have that $S \subset A_{\text{sa}}^k$ is dense. Consider the map $\Psi: S \rightarrow C(X, G_n^k/PU_n)$, as defined in 4.22. One checks that this map is continuous and open. Note that G_n^k/PU_n is a manifold of dimension $kn^2 - (n^2 - 1) = (k-1)n^2 + 1$. It follows from Proposition 4.6 that

$E(X, G_n^k/PU_n) \subset C(X, G_n^k/PU_n)$ is dense if and only if $\dim(X \times X) < (k-1)n^2 + 1$. But this follows from the assumption on $\dim(X)$, using $n \geq 2$ at the third step:

$$\dim(X \times X) \leq 2\dim(X) < 2(k-1)(2n-2) \leq (k-1)n^2 + 1.$$

Then $\Psi^{-1}(E(X, G_n^k/PU_n)) \subset S$ is dense. It follows from Proposition 4.16 that $\text{Gen}_k(A)_{\text{sa}} = \Psi^{-1}(E(X, G_n^k/PU_n))$, which shows $\text{gr}(A) \leq k-1$, as desired. \square

Lemma 4.24. *Let A be a separable n -homogeneous C^* -algebra. Set $X := \text{Prim}(A)$. If $\dim(X) < \infty$, then A is isomorphic to an ideal in a unital, separable n -homogeneous C^* -algebra B with $\dim(\text{Prim}(B)) = \dim(X)$.*

Proof. For $n = 1$ this is clear. For $n \geq 2$, this follows from Proposition 2.9 and Lemma 2.10 in [Phi07], since $\dim(X) < \infty$ implies that the M_n -bundle associated to A has finite type. \square

Corollary 4.25. *Let A be a separable n -homogeneous C^* -algebra. Set $X := \text{Prim}(A)$. If $n = 1$ (i.e., A is commutative), then*

$$\text{gr}(A) = \dim(X \times X).$$

If $n \geq 2$, then:

$$\text{gr}(A) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil.$$

Proof. For $n = 1$, this follows from Proposition 4.7. So assume $n \geq 2$. If A is unital, the formula follows from Theorem 4.23, so we may assume A is non-unital. We first show the inequality $\text{gr}(A) \geq \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil$. Given a compact subset $Y \subset X$, let $A(Y)$ denote the unital, separable, n -homogeneous quotient of A corresponding to Y . It follows from Theorem 4.23 and Proposition 2.12 that:

$$\left\lceil \frac{\dim(Y) + 1}{2n - 2} \right\rceil = \text{gr}(A(Y)) \leq \text{gr}(A).$$

The desired inequality follows, since $\dim(X)$ is equal to the maximum of $\dim(Y)$ when Y is running over the compact subsets of X .

The converse inequality is clear if $\dim(X) = \infty$, so assume $\dim(X) < \infty$. By Lemma 4.24, A is an ideal in a unital, separable n -homogeneous C^* -algebra B with $\dim(\text{Prim}(B)) = \dim(X)$. It follows from Theorem 2.14 and Theorem 4.23 that:

$$\text{gr}(A) \leq \text{gr}(B) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil,$$

as desired. \square

Lemma 4.26. *Let A, B be two separable homogeneous C^* -algebras. Then $\text{gr}(A \oplus B) = \max\{\text{gr}(A), \text{gr}(B)\}$.*

Proof. We have $\text{gr}(A), \text{gr}(B) \leq \text{gr}(A \oplus B)$ by Proposition 2.12. We need to show the converse inequality. Assume A is n -homogeneous, and B is m -homogeneous. Set $X := \text{Prim}(A)$, and $Y := \text{Prim}(B)$. If $n = m$, then $A \oplus B$ is n -homogeneous with $\text{Prim}(A \oplus B) = X \sqcup Y$. If $n = 1$, the result follows from Proposition 4.9.

If $n \geq 2$, then Corollary 4.25 shows:

$$\begin{aligned} \text{gr}(A \oplus B) &= \left\lceil \frac{\dim(X \sqcup Y) + 1}{2n - 2} \right\rceil \\ &= \max \left\{ \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil, \left\lceil \frac{\dim(Y) + 1}{2n - 2} \right\rceil \right\} \\ &= \max\{\text{gr}(A), \text{gr}(B)\}. \end{aligned}$$

For general C^* -algebras A, B one has $\text{Gen}_k(A \oplus B)_{\text{sa}} \subset \text{Gen}_k(A)_{\text{sa}} \oplus \text{Gen}_k(B)_{\text{sa}}$, and the inclusion might be strict. However, if A is n -homogeneous, and B is m -homogeneous with $n \neq m$, then no non-zero quotient of A is isomorphic to a quotient of B , and therefore:

$$\text{Gen}_k(A \oplus B)_{\text{sa}} = \text{Gen}_k(A)_{\text{sa}} \oplus \text{Gen}_k(B)_{\text{sa}},$$

from which the desired equality follows. \square

In the same way as Lemma 4.26, one proves the following result:

Corollary 4.27. *Let A_1, \dots, A_k be separable, homogeneous C^* -algebras. Then:*

$$\text{gr}\left(\bigoplus_i A_i\right) = \max_i \text{gr}(A_i).$$

Remark 4.28. Let A be a unital, separable, n -homogeneous C^* -algebra, $n \geq 2$, and set $X := \text{Prim}(A)$. It follows from Theorem 4.23 that the generator rank of A only depends on $\dim(X)$ (and n), but not on $\dim(X \times X)$. Thus, whether X is of basic or exceptional type does not matter for the computation of the generator rank of A .

Remark 4.29. Let $d \geq 1$ and $n \geq 2$, and set $A = C([0, 1]^d, M_n)$. Recall that we denote by $\text{gen}(A)$ the minimal number of self-adjoint generators for A . It follows from [Nag], [BE91] and Theorem 4.23 that:

$$\text{gen}(A) = \left\lceil \frac{d-1}{n^2} + 1 \right\rceil, \quad \text{rr}(A) = \left\lceil \frac{d}{2n-1} \right\rceil, \quad \text{gr}(A) = \left\lceil \frac{d+1}{2n-2} \right\rceil.$$

This shows that the generator rank is more closely connected to the real rank than to the minimal number of generators.

The generator problem for simple C^* -algebras asks whether every unital, separable, simple C^* -algebra A is singly generated, i.e., whether A contains a generating element. We might consider a strengthened version that asks if $\text{gr}(A) \leq 1$, i.e., whether the generating elements in A are dense. It follows from the work of Villadsen that this strengthened generator problem has a negative answer. Indeed, there exist simple AH-algebras of arbitrarily high real rank, see [Vil99]. Let A be such an AH-algebra with $\text{rr}(A) = \infty$. Then $\text{gr}(A) = \infty$, by Proposition 2.5.

Let A be a unital, separable, n -homogeneous C^* -algebra. Then A is an inductive limit of unital, separable, n -homogeneous C^* -algebras with finite-dimensional primitive ideal space. It follows from Theorem 4.23 and Proposition 2.13 that the tensor product of A with an infinite UHF-algebra has generator rank one. Using also Corollary 4.27, we may draw the following conclusion:

Corollary 4.30. *Let A be a unital, separable AH-algebra. Assume either that A is simple with slow dimension growth, or that A tensorially absorbs a UHF-algebra. Then $\text{gr}(A) \leq 1$, and so the generators of A form a generic subset.*

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